MINISTRY OF EDUCATION AND TRAINING

HANOI PEDAGOGICAL UNIVERSITY 2

TRAN VAN NGHI

EXISTENCE AND STABILITY FOR QUADRATIC PROGRAMMING PROBLEMS WITH NON-CONVEX OBJECTIVE FUNCTION

DOCTORAL DISSERTATION IN MATHEMATICS

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Supervisor: Assoc. Prof. Dr. Nguyen Nang Tam

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Confirmation

This dissertation has been written on the basis of my research works carried at Hanoi Pedagogical University 2, under the supervision of Assoc. Prof. Dr. Nguyen Nang Tam. The presented results have never been published by others.

The author

Tran Van Nghi

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Contents

Table of Notations

(P)	the optimization problem
LP	linear programming
NLP	nonlinear programming
QP	quadratic programming
$LCQP$	linearly constrained quadratic programming
$QCQP$	quadratically constrained quadratic programming
TRS	trust region subproblem
$ETRS$	extended trust region subproblem
VI	variational inequality
AVI	affine variational inequality
$EAVI$	extended affine variational inequality
$(QP(p))$	the QCQP problem depending on the parameter p
$\mathcal{F}(p)$	the feasible region of $(QP(p))$
$L(p)$	the local optimal solution set of $(QP(p))$
$IL(p)$	the isolated local optimal solution set of $(QP(p))$
$G(p)$	the global optimal solution set of $(QP(p))$
$S(p)$	the stationary solution set of $(QP(p))$
KKT	Karush-Kuhn-Tucker
$L(x, p, \lambda)$	the Lagrange function of $(QP(p))$
$\Lambda(\bar{x}, p)$	the set of all Lagrange multipliers corresponding to $\bar{x}$
$(VI(F, S))$	the VI depending on the function F and the S
$(VI(p))$	the VI depending on the parametric p
$(ET_m(w))$	the extended trust region subproblem depending on the parametric w

SCQ	Slater Constraint Qualification
$MFCQ$	Mangasarian-Fromovitz Constraint Qualification
$(MFRC)_{p^0}$	the Mangasarian-Fromovitz Regularity Condition under direction p^0
$LICQ$	Linear Independence Constraint Qualification
$I(\bar{x}, p)$	the active constraint index set of $(QP(p))$ at $\bar{x}$
$\mathbb{R}$	the real set
$\mathbb{R}^n$	the n-dimensional Euclidean space
$\mathbb{R}^{n\times n}_S$	the space of real symmetric $(n \times n)$–matrices
$\mathbb{R}^{n\times n}_{S^+}$	the set of positive semidefinite real symmetric $(n\times n)$–matrices
$\mathbb{R}^n_+$	$\{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_i \geq 0,\ i = 1, \ldots, n\}$
$x^T y$ or $\langle x, y\rangle$	the scalar product of vectors x, y
$\|x\|$	the Euclidean norm of a vector x
$\mathrm{dom}F$	effective domain of F
$\mathrm{gph}F$	graph of F
$\mathrm{Lim\,sup}$	limit in the sence Painlevé-Kuratowski
A^T	the transposed matrix of A
$\widehat{\mathcal{N}}(\bar{x}; \Omega)$	Fréchet normal cone of Ω at $\bar{x}$
$\mathcal{N}(\bar{x}; \Omega)$	Mordukhovich normal cone of Ω at $\bar{x}$
$\widehat{D}^*F(\bar{x}, \bar{y})(\cdot)$	Fréchet coderivative of F at $(\bar{x}, \bar{y})$
$D^*F(\bar{x}, \bar{y})(\cdot)$	Mordukhovich coderivative of F at $(\bar{x}, \bar{y})$
$x \xrightarrow{C} \bar{x}$	$x \to \bar{x}$ and $x \in C$
$\alpha \downarrow \bar{\alpha}$	$\alpha \to \bar{\alpha}$ and $\alpha \geq \bar{\alpha}$
$\alpha \uparrow \bar{\alpha}$	$\alpha \to \bar{\alpha}$ and $\alpha \leq \bar{\alpha}$
0^+C	the recession cone of C
$\varphi'(p, p^0)$	first-order directional derivative at p in direction p^0
$Null(Q)$	$\{x \in \mathbb{R}^n : Qx = 0\}$
$pos\{a, b\}$	$\{\theta a + \gamma b : \theta \geq 0, \gamma \geq 0\}$
S^*	$\{x \in \mathbb{R}^n : y^T x \geq 0\ \ \forall y \in S\}$

Introduction

Optimization concerns the analysis and the solution of problems in order to find the best elements in a given set. It is an important and very successful area of the applied mathematics. Applications of optimization are expanding and diverse. Among the most popular areas of application, we should mention as follows: engineering, statistics, economics, computer science, management sciences, and mathematics itself. Optimization problem arises in approximation theory, probability theory, structure design, chemical process control, routing in telecommunication networks, image reconstruction, experiment design, radiation therapy, asset valuation, portfolio management, supply chain management, facility location, and others.

Generally, an optimization problem (P) can be stated very simply as follows. We have a given set C and a real-valued function f on C. The problem is to find a point $\bar{x} \in C$ such that $f(\bar{x}) \leq f(x)$ for all $x \in C$. Then, C is called *the feasible set* or *the constraint region*, and the function f is called *the objective function.* Normally, C is defined by a system of equations and inequalities, which we call *constraints.* If $C = \mathbb{R}^n$ then we call the problem (P) to be the *unconstrained optimization* problem. We say that (P) is the *constrained problem* if C is a strict subset of the space $\mathbb{R}^n$ (i.e., $C \subset \mathbb{R}^n$ and $C \neq \mathbb{R}^n$). A feasible vector $\bar{x} \in C$ is called a *global solution* of (P) if $f(\bar{x}) \neq +\infty$ and $f(\bar{x}) \leq f(x)$ for all $x \in C$. We say that $\bar{x}$ is a *local solution* of (P) if $f(\bar{x}) \neq +\infty$ and there exists a neighborhood U of $\bar{x}$ such that $f(\bar{x}) \leq f(x)$ for all $x \in C \cap U$. The set of all the global solutions (resp., the local solutions) of (P) is denoted by $(G(P))$ (resp., $L(P)$).

The optimization theory includes various fields such as integer, stochastic, linear, nonlinear, convex, nonconvex, smooth, nonsmooth optimization, optimal control, semi-infinite programming, ect,. T here have been several main directions of research including: existence of solutions, optimality conditions, sensitivity analysis, duality theory and numerical methods.

The most popular constrained optimization problem is the *linear programming* (LP) problems, in which the objective function is a linear function and the constraint set is defined by finitely many linear equations and inequalities. If the objective function or some of the equations or inequalities defining the feasible set are nonlinear, the optimization problem is called the *nonlinear programming* (NLP) problem. In this case, the specific techniques and theoretical results of LP cannot be directly applied, and a more general approach is needed.

Quadratic programming (QP) problems constitute a special class of NLP problems. Numerous problems in real world applications, including problems in planning and scheduling, economies of scale, engineering design, and control are naturally expressed as QP problems. One also uses QP problems in order to approximate NLP problems. The importance of QP was presented by Floudas and Visweswaran [33].

Many important research results for *linearly constrained quadratic programming* (LCQP) problems can be found in Lee et al. [56] and the references cited therein. Since the finite dimensional LCQP problems have been rather comprehensively investigated, several authors are now interested in studying *quadratically constrained non-convex quadratic programming* (QCQP) problems.

The study of QCQP problems originated in 1951 by Kuhn and Tucker [55], if not earlier. These problems have been of great interests to the researchers in theory and practice. Besides the theoretical importance, QCQP problems are of wide applications. In *numerical optimization*, at each iteration of the trust region method, a QP problem with one elliptic constraint is solved as a subproblem in order to find a moving direction. This subproblem is a special case of QCQP and is known as the *trust region subproblem (TRS)*. In *binary integer programming* problems, the integer requirements can be formulated as quadratic constraints. In *statistics*, the linear regression model minimizes an unconstrained quadratic function which is a special case of QCQP.

On *qualitative properties* of QCQP problems, one often concerns solution existence, optimal conditions, sensitivity analysis and stability.

The solution existence of QP problems is one of the most important issues. In 1956, Frank and Wolfe [34] extended the fundamental existence of linear programming by proving that an arbitrary quadratic function f attains its minimum over a nonempty convex polyhedral set C provided that f is bounded from below over C (called *Frank-Wolfe Theorem*). From then to now, there have been some other proofs for this theorem and its extended versions. Belousov [12, Chapter II, Section 4, Theorem 13] and Terlaky [105] proved the following result: A QP problem has a solution if its objective function is convex and bounded below over a nonempty constraint set defined by convex quadratic functions. Detailed proofs of this result can be found in [13,66]. In 1999, Luo and Zhang [66, Theorem 2] proved that a QP problem has a solution if its objective function is bounded below over a nonempty constraint set defined by a convex quadratic function and linear constraint functions. They also showed [66, Example 2, p. 94] that there exists a nonconvex QP problem in $\mathbb{R}^4$ with two convex quadratic constraints whose objective function is bounded from below over a nonempty constraint set, which has no solutions. Belousov and Klatte [13, p. 45] observed that the effect of nonconvexity of the objective function can be seen in $\mathbb{R}^3$. Bertsekas and Tseng [14] proved the solution existence of a QP problem when all the asymptotic directions of constraint set are retractive local horizon directions and the objective function is bounded below constraint set. Tuy and Tuan [106] established some important results on the solution existence for nonconvex QP problems. Given a quadratic function and a convex quadratic constraint set, verifying whether the function is bounded from below on the set is a rather difficult task. Eaves [31] discussed another fundamental existence theorem (called *Eaves Theorem*) for LCQP problems which gives us a tool for dealing with the task. Eaves Theorem presented verifiable necessary and sufficient conditions for the solution existence of LCQP problems. By using the concept of recession cone in convex analysis, Lee et al. [61] established an Eaves type Theorem for convex QCQP problems. Up to now, many researchers have been studying sufficient conditions for the solution existence of a nonconvex QP problem whose constraint set is defined by finitely many quadratic inequalities.

Stability for parametric QCQP problems plays an important role because they can be used for analyzing algorithms for solving this problem. For convex QP problems, Best et al. [15, 16] obtained some results on the continuity and differentiability of the optimal value function; some continuity and/or differentiability properties of the global optimal solution map have been discussed (see, for example, [6, 10, 15, 26, 27, 43, 88]). For nonconvex LCQP problems, the continuity for the global optimal solution map, stationary solution map and the optimal value function have been investigated in details in [56] and the references therein. For TRSs, Lee et al. [61] investigated the case where the linear part or the quadratic form of the objective function is perturbed and obtained necessary and sufficient conditions for the upper/lower semicontinuity of the stationary solution map and the global optimal solution map, explicit formulas for computing the directional derivative and the Fréchet derivative of the optimal value function. Lee and Yen [62] estimated the Mordukhovich coderivative and conditions for the local Lipschitz-like property of the stationary solution map in parametric TRS. Since QP is a class of nonlinear optimization problems, the results in nonlinear optimization can be applied to convex and nonconvex QP problems. Differential properties of the marginal function and of the global optimal solution map in mathematical programming were investigated by Gauvin and Dubeau [36]. Continuity and Lipschitzian properties of the optimal value function have been studied in [10, 91]. Auslender and Cominetti [5] considered first and second-order sensitivity analysis of NLP under directional constraint qualification conditions. In [72], Minchenko and Tarakanov discussed directional derivatives of the optimal value functions under the assumption of the calmness of global optimal solution mapping. Lipschitzian continuity of the optimal value function was presented by Dempe and Mehlitz [28]. Some similar topics related to Lipschitzian stability have been investigated in [37, 64, 71, 94] and the references given there. A survey of recent results on stability of NLP problems was given by Bonnans and Shapiro [18]. In which, many interesting results can be applied for QP. However, the special structure of QP problems allows one to have deeper and more comprehensive results on stability for QCQP.

This dissertation gives new results on *the existence and stability for quadratic programming problems with non-convex objective function.* By using the special structure of quadratic forms, the recession cone and some advanced tools of variational analysis, we propose conditions for the solution existence and investigate in details the stability for QCQP problems. The specific techniques and theoretical results for LCQP and TRS cannot be directly applied, and a more general approach is used. Among our proposed assumptions, there are some weaker than ones used in the cited works (applied for QP). We also generalize some stability results from the case of polyhedral convex constraint set to the case of constraint set defined by finitely many convex quadratic functions.

The dissertation has four chapters and a list of references.

Chapter 1 presents conditions for the solution existence of QCQP problems through a Frank-Wolfe type Theorem and an Eaves type Theorem.

Chapter 2 investigates the continuity of the global, local and stationary solution maps of parametric QCQP problems by using the obtained results on solution existence.

Chapter 3 characterizes the continuity, Lipschitzian continuity and directional differentiability of the optimal value function under weaker assumptions in comparison with results which are implied from general theory.

Chapter 4 devotes detailed discussion to the stability for parametric extended trust region subproblem (ETRS). We estimate the Mordukhovich coderivative of the stationary solution map and use the obtained results to investigate Lipschitzian stability for parametric ETRS.

The dissertation is written on the basis of the paper [75] in *Acta Math. Vietnam.*, the paper [102] in *Optim. Lett.*, the paper [76] in *Taiwanese J. Math.*, the paper [77] in *Optimization*, and preprints [78] and [79], which have been submitted.

The results of this dissertation were presented at International Workshop on New Trends in Optimization and Variational Analysis for Applications (Quynhon, December 7–10, 2016); The 14$^{\text{th}}$ Workshop on Optimization and Scientific Computing (Bavi, April 21–23, 2016); The 5$^{\text{th}}$ National Workshop of young researchers in teacher training university (Vinhphuc, May 23–24, 2015); Scientific Conference at Hanoi Pedagogical University 2 (HPU2) (Vinhphuc, November 14, 2015); at the seminar of Department of Mathematics, HPU2 and at the seminar of the Department of Numerical Analysis and Scientific Computing, Institute of Mathematics, Vietnam Academy of Science and Technology.

Chapter 1

Existence of solutions

The aim of this chapter is to investigate the existence of solutions of QCQP problems. The QCQP problem is stated in Section 1.1. Sections 1.2-1.3 present a Frank-Wolfe type theorem and an Eaves type theorem for solution existence.

The presentation given below comes from the results in [102].

1.1. Problem statement

Let $\mathbb{R}^n$ be n-dimensional Euclidean space equipped with the standard scalar product and the Euclidean norm, $\mathbb{R}^{n\times n}_S$ be the space of real symmetric $(n \times n)$–matrices equipped with the matrix norm induced by the vector norm in $\mathbb{R}^n$ and $\mathbb{R}^{n\times n}_{S^+}$ be the set of positive semidefinite real symmetric $(n \times n)$–matrices. Let

$$\mathbb{P} := \mathbb{R}^{n\times n}_S \times \mathbb{R}^n \times \underbrace{(\mathbb{R}^{n\times n}_{S^+} \times \mathbb{R}^n \times \mathbb{R}) \times \ldots \times (\mathbb{R}^{n\times n}_{S^+} \times \mathbb{R}^n \times \mathbb{R})}_{m\ times} \subset \mathbb{R}^s$$

with $s = (m + 1)(n^2 + n + 1) - 1$. The scalar product of vectors x, y and the Euclidean norm of a vector x in a finite-dimensional Euclidean space are denoted, respectively, by $x^T y$ (or $\langle x, y\rangle$) and $\|x\|$, where the superscript ^{T} denotes the transposition. Vectors in Euclidean spaces are interpreted as columns of real numbers. The notation $x \geq y$ (resp.,

$x > y$) means that every component of x is greater or equal (resp., greater) the corresponding component of y. For $D \in \mathbb{R}^{n\times n}_S$, we define

$$\|D\| = \max\{\|Dx\| : x \in \mathbb{R}^n, \|x\| \leq 1\}.$$

The norm in the product space $X_1 \times \ldots \times X_k$ of the normed spaces $X_1, \ldots, X_k$ is set to be

$$\|(x_1, \ldots, x_k)\| = (\|x_1\|^2 + \ldots + \|x_k\|^2)^{\frac{1}{2}}.$$

Let us consider the following nonconvex quadratic programming problem under convex quadratic constraints

$$\begin{aligned} &\min f(x,p) = \tfrac{1}{2}x^TQx + q^Tx \\ &s.t.\ x \in \mathbb{R}^n : g_i(x,p) = \tfrac{1}{2}x^TQ_ix + q_i^Tx + c_i \leq 0, \\ &\qquad\qquad\qquad\qquad\qquad\qquad\quad i = 1, \ldots, m, \end{aligned} \tag{$QP(p)$}$$

depending on the parameter $p = (Q, q, Q_1, q_1, c_1, \ldots, Q_m, q_m, c_m) \in \mathbb{P}$. For each $i \in \{1, \ldots, m\}$, by the fact that $Q_i \in \mathbb{R}^{n\times n}_{S^+}$, we have $g_i(x,p)$ is a convex quadratic function.

The *feasible region*, the *local optimal solution set* and the *global optimal solution set* of $(QP(p))$ will be denoted by $\mathcal{F}(p)$, $L(p)$, and $G(p)$, respectively.

The *recession cone* (see, for instance, [18, p. 33]) of $\mathcal{F}(p) \neq \emptyset$ is defined by

$$0^+\mathcal{F}(p) = \{v \in \mathbb{R}^n : x + tv \in \mathcal{F}(p)\ \ \forall x \in \mathcal{F}(p)\ \ \forall t \geq 0\}.$$

According to [49, Lemma 1.1], we obtain that

$$0^+\mathcal{F}(p) = \{v \in \mathbb{R}^n : Q_iv = 0, q_i^Tv \leq 0,\ \forall i = 1, \ldots, m\}. \tag{1.1}$$

The function

$$\varphi : \mathbb{P} \longrightarrow \mathbb{R} \cup \{\pm\infty\}$$

defined by

$$\varphi(p) = \begin{cases} \inf\{f(x,p) : x \in \mathcal{F}(p)\} & \text{if } \mathcal{F}(p) \neq \emptyset; \\ +\infty & \text{if } \mathcal{F}(p) = \emptyset, \end{cases}$$

is called *the optimal value function* of the parametric problem $(QP(p))$.

1.2. A Frank-Wolfe type theorem

In this section, we present a sufficient condition for the solution existence of a nonconvex QP problem whose constraint set is defined by finitely many convex quadratic inequalities $(QP(p))$. The obtained result complements and develops the corresponding published result of Luo and Zhang [66, Theorem 2].

Fix $p \in \mathbb{P}$ and let

$$I = \{1, \ldots, m\},\ I_0 = \{i \in I :\ Q_i = 0\},\ I_1 = \{i \in I :\ Q_i \neq 0\} = I \setminus I_0.$$

Before stating the main results, we need the following technical lemma.

Lemma 1.1. *Assume that $\{x^k\} \subset \mathcal{F}(p)$ such that $x^k \neq 0$, $\|x^k\| \to \infty$ and $\|x^k\|^{-1}x^k \to \bar{v}$. Then, $\bar{v} \in 0^+\mathcal{F}(p)$.*

Proof. Since $x^k \in \mathcal{F}(p)$, we have

$$g_i(x^k, p) := \frac{1}{2}(x^k)^T Q_i x^k + q_i^T x^k + c_i \leq 0,\ i = 1, \ldots, m. \tag{1.2}$$

Since $Q_i,\ i = 1, \ldots, m$, are positive semidefinite, by (1.2) we obtain that

$$q_i^T x^k + c_i \leq 0,\ i = 1, \ldots, m. \tag{1.3}$$

Dividing both sides of the inequality (1.3) by $\|x^k\|$ and letting $k \to \infty$ yields

$$q_i^T \bar{v} \leq 0,\ i = 1, \ldots, m. \tag{1.4}$$

Multiplying the inequality in (1.2) by $\|x^k\|^{-2}$ and letting $k \to \infty$ yields $\bar{v}^T Q_i \bar{v} \leq 0,\ i = 1, \ldots, m$. From the fact that, for each $i = 1,\ \ldots, m$, Q_i are positive semidefinite it follows that $\bar{v}^T Q_i \bar{v} = 0$; that is, $x = \bar{v}$ is a solution of the unconstrained optimization problem

$$\min\{\varphi(x) = \frac{1}{2}x^T Q_i x : x \in \mathbb{R}^n\}.$$

Combining this with the Fermat rule yields

$$Q_i \bar{v} = 0,\ i = 1, \ldots, m. \tag{1.5}$$

By (1.1), (1.4), and (1.5), we obtain $\bar{v} \in 0^+\mathcal{F}(p)$. □

The following result is a generalization of Frank-Wolfe Theorem.

Theorem 1.1. *Consider the problem $(QP(p))$. Assume that $\mathcal{F}(p)$ is nonempty, $f(x,p)$ is bounded from below over $\mathcal{F}(p)$ and one of the following conditions is satisfied:*

(A_1) *The set I_1 consists of at most one element;*

(A_2) *For each $v \in 0^+\mathcal{F}(p)$, if $v^T Q v = 0$ then $q_i^T v = 0$ for all $i \in I_1$.*

Then, $(QP(p))$ has a solution.

Proof. Assume that (A_1) holds. From [66, Theorem 2] it follows that $(QP(p))$ has a solution.

We now assume that (A_2) holds. Let $f^* = \inf\{f(x,p) :\ x \in \mathcal{F}(p)\}$. For each positive integer k, let $S_k = \{x \in \mathcal{F}(p) :\ \ f(x,p) \leq f^* + \frac{1}{k}\}$. Since $f^* > -\infty$, S_k is nonempty and closed. Let x^k be the smallest norm element in S_k. Then,

$$g_i(x^k,p) = \frac{1}{2}(x^k)^T Q_i x^k + q_i^T x^k + c_i \leq 0 \quad \forall i = 1, ..., m,\ \ \forall k \geq 1 \tag{1.6}$$

and

$$f(x^k,p) = \frac{1}{2}(x^k)^T Q x^k + q^T x^k \leq f^* + \frac{1}{k} \quad \forall k \geq 1. \tag{1.7}$$

We first show that $\{x^k\}$ is bounded. Indeed, suppose that $\{x^k\}$ is unbounded. Without loss of generality we may assume that $\|x^k\| \to \infty$

as $k \to \infty$, $\|x^k\| \neq 0$ for all k and $\|x^k\|^{-1}x^k \to \bar{v}$ with $\|\bar{v}\| = 1$. By Lemma 1.1, we have $\bar{v} \in (0^+\mathcal{F}(p)) \setminus \{0\}$. Multiplying the inequality $\frac{1}{2}(x^k)^T Q x^k + q^T x^k \leq f^* + \frac{1}{k}$ in (1.7) by $\|x^k\|^{-2}$ and passing to the limit as $k \to \infty$, we obtain $\bar{v}^T Q \bar{v} \leq 0$. If $\bar{v}^T Q \bar{v} < 0$ then

$$f(x^k + t\bar{v}, p) = f(x^k, p) + \frac{t^2}{2}\bar{v}^T Q \bar{v} + t(Qx^k + q)^T \bar{v} \to -\infty \text{ as } t \to \infty.$$

Hence

$$\bar{v}^T Q \bar{v} = 0. \tag{1.8}$$

If there exists k such that $(Qx^k + q)^T \bar{v} < 0$ then $x^k + t\bar{v} \in \mathcal{F}(p)$ for all $t > 0$. By (1.8) we have

$$f(x^k + t\bar{v}, p) = f(x^k, p) + t(Qx^k + q)^T \bar{v} \to -\infty \text{ as } t \to \infty,$$

contradicts the assumption that $f(x, p)$ is bounded from below over $\mathcal{F}(p)$. Hence

$$(Qx^k + q)^T \bar{v} \geq 0 \tag{1.9}$$

for every k.

Now we show that there exists k_0 such that $x^k - t\bar{v} \in \mathcal{F}(p)$ for all $k \geq k_0$ and for all $t > 0$ small enough. To do this, recall that $I = \{1, ..., m\}$, $I_1 = \{i : \; Q_i \neq 0\} \subset I$ and $I_0 = I \setminus I_1 = \{i : \; Q_i = 0\}$. By (1.6), we see that, for each i, $\{g_i(x^k, p)\}$ is bounded from above. Therefore there exists a subsequence $\{k^s\}$ of $\{k\}$ such that all the limits $\lim\limits_{s\to\infty} g_i(x^{k^s}, p)$ exist, $i = 1, ..., m$. Let us assume, without loss of generality, that $\{k^s\} \equiv \{k\}$. Denote:

$$J_1 := \{i \in I_0 : \; \lim_{k\to\infty} g_i(x^k, p) = 0\} = \{i \in I_0 : \; \lim_{k\to\infty} (q_i^T x^k + c_i) = 0\};$$

$$J_2 := \{i \in I_0 : \; \lim_{k\to\infty} g_i(x^k, p) < 0\} = \{i \in I_0 : \; \lim_{k\to\infty} (q_i^T x^k + c_i) < 0\}.$$

Since $\lim\limits_{k\to\infty} (q_i^T x^k + c_i) = 0$ for all $i \in J_1$, we can check that

$$q_i^T \bar{v} = 0 \;\; \forall i \in J_1.$$

By (1.8) and the assumption (A_2), we have $q_i^T\bar{v} = 0 \ \ \forall i \in I_1$. Hence

$$q_i^T\bar{v} = 0 \ \ \forall i \in J_1 \cup I_1. \tag{1.10}$$

For each $i \in J_1 \cup I_1$, from (1.10) and Lemma 1.1 it follows that

$$\begin{aligned} g_i(x^k - t\bar{v}, p) &= \tfrac{1}{2}(x^k - t\bar{v})^T Q_i (x^k - t\bar{v}) + q_i^T(x^k - t\bar{v}) + c_i \\ &= \tfrac{1}{2}(x^k)^T Q_i x^k + q_i^T x^k + c_i \\ &= g_i(x^k, p) \le 0. \end{aligned} \tag{1.11}$$

Since $\lim\limits_{k\to\infty} g_i(x^k, p) = \lim\limits_{k\to\infty}(q_i^T x^k + c_i) < 0$ for any $i \in J_2$, there exists $\varepsilon > 0$ such that

$$\lim_{k\to\infty} g_i(x^k, p) = \lim_{k\to\infty}(q_i^T x^k + c_i) \le -\varepsilon \quad \ \forall i \in J_2.$$

For each $i \in J_2$, there exists $k_1 > 0$ such that

$$g_i(x^k, p) = q_i^T x^k + c_i \le -\frac{\varepsilon}{2} \quad \ \forall k \ge k_1.$$

Fix $k \ge k_1$ and choose $\delta_{k,i} > 0$ so that

$$tq_i^T\bar{v} \ge -\frac{\varepsilon}{2}$$

for all $t \in (0, \delta_{k,i})$. Then,

$$\begin{aligned} g_i(x^k - t\bar{v}, p) &= q^T(x^k - t\bar{v}) + c_i \\ &\le q^T x^k + c_i - tq_i^T\bar{v} \\ &\le -\tfrac{\varepsilon}{2} - tq_i^T\bar{v} \\ &\le \ 0 \qquad \forall i \in J_2. \end{aligned} \tag{1.12}$$

Let $\delta_k := \min\{\delta_{k,i} : \ i \in J_2\}$. From (1.11) and (1.12) it follows that

$$g_i(x^k - t\bar{v}, p) \le 0 \ \ \forall t \in (0, \delta_k) \quad \forall i = 1, ..., m.$$

This means

$$x^k - t\bar{v} \in \mathcal{F}(p) \quad \forall k \ge k_1 \quad \ \forall t \in (0, \delta_k). \tag{1.13}$$

By (1.8) and (1.9), we have

$$\begin{aligned} f(x^k - t\bar{v}, p) &= \tfrac{1}{2}(x^k - t\bar{v})^T Q (x^k - t\bar{v}) + q^T(x^k - t\bar{v}) \\ &= f(x^k, p) + t^2\bar{v}^T Q\bar{v} - t(Qx^k + q)^T\bar{v} \\ &\le f(x^k, p). \end{aligned} \tag{1.14}$$

Combining (1.13) with (1.14) yields

$$x^k - t\bar{v} \in S_k \quad \forall k \geq k_1, \forall t \in (0, \delta_k). \tag{1.15}$$

Since $\bar{v}^T\bar{v} = 1$ and $\|x^k\|^{-1}x^k \to \bar{v}$, there exists $k_2 \geq k_1$ such that

$$(x^k)^T\bar{v} > 0 \quad \forall k \geq k_2.$$

Consequently, there exists $\gamma > 0$ such that

$$\begin{aligned} \|x^k - t\bar{v}\|^2 &= \|x^k\|^2 - 2t(x^k)^T\bar{v} + t^2\|\bar{v}\|^2 \\ &< \|x^k\|^2 \quad \forall t \in (0, \gamma). \end{aligned} \tag{1.16}$$

Let $\delta := \min\{\delta_k, \gamma\}$. Then, by (1.15) and (1.16), we have

$$x^k - t\bar{v} \in S_k \quad \text{and } \|x^k - t\bar{v}\| < \|x^k\| \quad \forall k \geq k_2, \quad \forall t \in (0, \delta).$$

This contradicts the fact that x^k is the smallest norm element in S_k. Therefore, we conclude that $\|x^k\|$ must be bounded.

Since $\|x^k\|$ is bounded, it has a convergent subsequence. Without loss of generality, we can assume that $x^k \to \overline{x}$ as $k \to \infty$. Since $\mathcal{F}(p)$ is closed and $x^k \in \mathcal{F}(p)$, we have $\bar{x} \in \mathcal{F}(p)$. From (1.7),

$$\begin{aligned} f(\bar{x}, p) = \frac{1}{2}\bar{x}^TQ\bar{x} + q^T\bar{x} &= \lim_{k\to\infty}\Big(\frac{1}{2}(x^k)^TQx^k + q^Tx^k\Big) \\ &\leq \lim_{k\to\infty}\big(f^* + \frac{1}{k}\big) \leq f^*. \end{aligned}$$

It follows that $\bar{x}$ is a solution of $(QP(p))$. The proof is complete. □

We obtain some important consequences of Theorem 1.1.

Corollary 1.1. (Frank-Wolfe Theorem) *Consider the quadratic programming problem under linear constraints* (LCQP) *(i.e.,* $(QP(p))$ *with* $Q_i = 0$ *for all* $i = 1, ..., m$*). Assume that* $f(x, p)$ *is bounded from below over nonempty* $\mathcal{F}(p)$*. Then, the problem* (LCQP) *has a solution.*

Proof. Since $Q_i = 0$ for all $i = 1, ..., m$, we have $I_1 = \emptyset$. Hence the condition (A_2) is automatically satisfied and the corollary follows. □

Corollary 1.2. *Assume that the function* $f(x,p) = \frac{1}{2}x^TQx + q^Tx$ *is bounded from below over* $\mathbb{R}^n$. *Then, there exists a* $x^* \in \mathbb{R}^n$ *such that* $f(x^*,p) \leq f(x,p)$ *for all* $x \in \mathbb{R}^n$.

Proof. Consider $(QP(p))$ with $Q_i = 0$, $q_i = 0$ and $c_i = 0$ for every $i = 1, ..., m$. Then, $\mathcal{F}(p) = \mathbb{R}^n$ and it is clear that the condition (A_2) is satisfied. The conclusion follows. □

Corollary 1.3. *Consider the problem* $(QP(p))$. *If* $\mathcal{F}(p)$ *is nonempty and* $v^TQv > 0$ *for every nonzero vector* $v \in 0^+\mathcal{F}(p)$ *then* $G(p)$ *is a nonempty compact set.*

Proof. Suppose that, contrary to our claim, $\mathcal{F}(p) \neq \emptyset$, $v^TQv > 0$ for all $v \in (0^+\mathcal{F}(p)) \setminus 0$ and $G(p) = \emptyset$ for some $(c_1, \ldots, c_m) \in \mathbb{R}^m$. By Theorem 1.1, there exists $x^k \in \mathcal{F}(p)$ such that $f(x^k, p) \to -\infty$. Then, $\|x^k\| \to \infty$ as $k \to \infty$. Without loss of generality, we assume that $x^k/\|x^k\| \to \bar{v} \in R^n$ and $f(x^k, p) < 0$, that is $\frac{1}{2}(x^k)^TQx^k + q^Tx^k < 0$. Dividing both sides of the later by $\|x^k\|^2$ and letting $k \to \infty$, we get $\bar{v}^TQ\bar{v} \leq 0$. Since $x^k \in \mathcal{F}(p)$, we have $g_i(x^k, p) \leq 0$, $i = 1, \ldots, m$. From Lemma 1.1 it follows that $\bar{v} \in (0^+\mathcal{F}(p)) \setminus 0$ and $\bar{v}^TQ\bar{v} \leq 0$. This contradicts the assumption. Hence $G(p) \neq \emptyset$.

Suppose that $G(p)$ is unbounded for some $(c_1, \ldots, c_m) \in \mathbb{R}^m$. Then, there exists a sequence $\{y^k\} \subset G(p)$ such that $\|y^k\| \to \infty$ as $k \to \infty$. By passing to a subsequence if necessary, we may assume that $y^k/\|y^k\| \to \bar{w}$ for some $\bar{w} \in \mathbb{R}^n \setminus \{0\}$. From $y^k \in \mathcal{F}(p)$ it follows $g_i(y^k, p) \leq 0$, $i = 1, \ldots, m$. By Lemma 1.1, we obtain$\bar{w} \in (0^+\mathcal{F}(p)) \setminus \{0\}$. This and $\bar{w}^TQ\bar{w} \leq 0$ contradict the assumption. Hence $G(p)$ is bounded. Since the closedness of $G(p)$ is obvious, we obtain that $G(p)$ is compact. □

The following example illustrates an application of Theorem 1.1.

Example 1.1. *Consider the problem* $(QP(p))$ *with*

$$p = (Q, q, Q_1, q_1, c_1, Q_2, q_2, c_2),$$

where

$$Q = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 2 \\ 0 & 2 & -2 \end{pmatrix}, \ q = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}, \ Q_1 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$q_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \ c_1 = -2, \ Q_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & -2 \\ 0 & -2 & 2 \end{pmatrix}, \ q_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \ c_2 = 0.$$

This problem can be rewritten as follows

$$\min\{f(x,p) = -x_1^2 - x_2^2 - x_3^2 + 2x_2x_3 - x_1 - x_2 + x_3 : x \in \mathcal{F}(p)\},$$

where

$$\mathcal{F}(p) = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : \ x_1^2 + x_1 - 2 \leq 0, \ x_2^2 + x_3^2 - 2x_2x_3 + x_2 - x_3 \leq 0\}.$$

Clearly, $\mathcal{F}(p) \neq \emptyset$. *One has*

$$f(x,p) = -(x_1^2 + x_1 - 2) - (x_2^2 + x_3^2 - 2x_2x_3 + x_2 - x_3) - 2 \geq -2 \ \forall x \in \mathcal{F}(p).$$

Hence $f(x,p)$ *is bounded from below over* $\mathcal{F}(p)$. *It can be verified that*

$$0^+\mathcal{F}(p) = \{(v_1, v_2, v_3) \in \mathbb{R}^3 : v_1 = 0, v_2 = v_3\}.$$

For each $v = (v_1, v_2, v_3) \in 0^+\mathcal{F}(p)$, *we have*

$$v^TQv = -2v_1^2 - 2v_2^2 - 2v_3^2 + 4v_2v_3 = 0,$$

$$q_1^Tv = 2v_1 = 0,$$

$$q_2^Tv = v_2 - v_3 = 0.$$

It follows that (A_2) *holds. By Theorem 1.1, this problem has a solution.*

The following example, which has been given by Belousov and Klatte [13, p.45], shows that Theorem 1.1 is not true if (A_2) is omitted.

Example 1.2. *Let us consider the problem $(QP(p))$ with $m = 2$, $n = 3$ and*

$$Q = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -2 \\ 0 & -2 & 0 \end{pmatrix}, \quad q = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}, \qquad Q_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$q_1 = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}, \quad c_1 = 0, Q_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}, q_2 = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}, c_2 = -1.$$

This problem is rewritten as follows

$$\min\{f(x, p) = -2x_2x_3 + 2x_1 : x \in \mathcal{F}(p)\},$$

where $\mathcal{F}(p) = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_2^2 - x_1 \le 0; x_3^2 - x_1 - 1 \le 0\}$. Since $(0, 0, 0) \in \mathcal{F}(p)$, $\mathcal{F}(p)$ is nonempty. It can be verified that

$$0^+\mathcal{F}(p) = \{(v_1, v_2, v_3) \in \mathbb{R}^3 : v_1 \ge 0, v_2 = v_3 = 0\}.$$

There exists $v = (1, 0, 0) \in 0^+\mathcal{F}(p)$ such that $v^TQv = -4v_2v_3 = 0$, but

$$q_1^Tv = q_2^Tv = -1 < 0.$$

Hence (A_2) is not satisfied. For any $x \in \mathcal{F}(p)$, one has

$$f(x, p) = -(x_2^2 - x_1) - (x_3^2 - x_1 - 1) + (x_2 - x_3)^2 - 1 > -1 \ \forall x \in \mathcal{F}(p).$$

Thus $f(x, p)$ is bounded from below over $\mathcal{F}(p)$. On the other hand, for the sequence $\{x^k = (k, \sqrt{k}, \sqrt{k+1})\} \subset \mathcal{F}(p)$, we have

$$f(x^k, p) \to -1 \text{ as } k \to +\infty.$$

Hence this problem has no solution.

Remark 1.1. *Luo and Zhang [66] considered the problem that has its polyhedral constraints explicitly stated: $Ax \le b$. Then, they proved [66, Theorem 3] that the given problem has a solution if the objective function*

f is quasi-convex (see, for instance, [38, Definition 2.10.1]) over the polyhedral set $\{x : Ax \leq b\}$.

To apply the latter result to $(QP(p))$, we need to show that there exists a polyhedral set Δ containing $\mathcal{F}(p)$ such that f is quasi-convex over Δ.

In Example 1.2, for any polyhedral set Δ containing $\mathcal{F}(p)$, f is not quasi-convex over Δ. Indeed, take $\bar{x} = (1, 1, 1)$, $\bar{y} = (1/2, 0, 0)$. Then, $\bar{x}, \bar{y} \in \mathcal{F}(p)$ and $f(\bar{x}, p) = 0$, $f(\bar{y}, p) = 1$. For each $t \in [0, 1]$, we have $f(t\bar{x} + (1-t)\bar{y}, p) = -2t^2 + t + 1$. By choosing $t_0 = 1/4$, we get $f(t_0\bar{x} + (1-t_0)\bar{y}, p) = 9/8$ and so

$$f(t_0\bar{x} + (1-t_0)\bar{y}, p) > max\{f(\bar{x}, p), f(\bar{y}, p)\},$$

which proves that f is not quasi-convex over $\mathcal{F}(p)$. It implies that f is not quasi-convex over Δ. Because of this reason, Theorem 3 in [66] does not work for this example.

Example 1.2 also shows that the quasi-convexity of the objective function over the polyhedral set $\{x : Ax \leq b\}$ cannot be dropped from the assumptions of Theorem 3 in [66].

1.3. An Eaves type theorem

Eaves [31] presented another fundamental existence theorem for LCQP problems (called *Eaves Theorem*) which gives us a tool for checking the boundedness from below of the object function on constraints set.

Unlike the case of LCQP, Eaves type necessary conditions for the solution existence of $(QP(p))$ do not coincide with the sufficient ones. The following result is a generalization of Eaves Theorem.

Theorem 1.2. *Consider $(QP(p))$ and assume that $\mathcal{F}(p)$ is nonempty. The following statements are valid:*

a) If $(QP(p))$ has a solution, then

i) $v^TQv \geq 0 \;\; \forall v \in 0^+\mathcal{F}(p),$ (1.17)

ii) $(Qx+q)^Tv \geq 0 \;\; \forall x \in \mathcal{F}(p) \forall v \in \{u \in 0^+\mathcal{F}(p) : u^TQu = 0\};$ (1.18)

b) If (1.17), (1.18) *and* (A_2) *hold, then* $(QP(p))$ *has a solution.*

Proof. a) Suppose that $(QP(p))$ has a solution $\bar{x}$. To obtain (1.17), let $v \in 0^+\mathcal{F}(p)$. Since $\bar{x} \in \mathcal{F}(p)$, we have $\bar{x}+tv \in \mathcal{F}(p)$ for every $t \geq 0$. Hence $f(\bar{x}+tv,p) \geq f(\bar{x},p)$ for every $t \geq 0$. It follows that $\frac{1}{2}t^2v^TQv+t(Q\bar{x}+c)^Tv \geq 0$ for every $t \geq 0$; hence $v^TQv \geq 0$. This shows that (1.17) is satisfied.

Now suppose that there are given any $v \in 0^+\mathcal{F}(p)$ with $v^TQv = 0$ and $x \in \mathcal{F}(p)$. Since $x+tv \in \mathcal{F}(p)$ for every $t \geq 0$ and $\bar{x}$ is a solution of $(QP(p))$, we have $f(x+tv,p) \geq f(\bar{x},p)$ for every $t \geq 0$. From this and the condition $v^TQv = 0$ we deduce that $t(Qx+q)^Tv+\frac{1}{2}x^TQx+q^Tx \geq f(\bar{x},p)$ for every $t \geq 0$. This implies $(Qx+q)^Tv \geq 0$. Hence (1.18) is satisfied.

b) To prove that $(QP(p))$ has a solution under additional assumption (A_2), by Theorem 1.1 it suffices to verify that f is bounded from below over $\mathcal{F}(p)$.

Define $f^* = \inf\{f(x,p) : x \in \mathcal{F}(p)\}$. As $\mathcal{F}(p) \neq \emptyset$, we have $f^* \neq +\infty$. Hence we only need to show that $f^* > -\infty$. To obtain a contradiction, suppose that $f^* = -\infty$. Then, there exists a sequence $\{y^k\} \subset \mathcal{F}(p)$ such that $f(y^k) \to -\infty$. There is no loss of generality in assuming that $\|y^k\| \to \infty$ as $k \to \infty$ and $f(y^k) \leq \frac{1}{k}$.

Let $S_k = \{x \in \mathcal{F}(p) : \; f(x,p) \leq f(y^k,p)\}$. We have $y^k \in S_k$, so S_k is nonempty and closed. Let x^k be the smallest norm element in S_k. Since $f(x^k,p) \leq f(y^k,p)$ and $f(y^k,p) \to -\infty$, we have $f(x^k,p) \to -\infty$. There is no loss of generality in assuming that $\|x^k\| \to \infty$ as $k \to \infty$. Note that

$$g_i(x^k,p) = \frac{1}{2}(x^k)^TQ_ix^k + q_i^Tx^k + c_i \leq 0, \quad \forall i = 1,...,m, \;\; \forall k \geq 1. \tag{1.19}$$

Without loss of generality we may assume that $\|x^k\| \neq 0$ for all k, $\frac{x^k}{\|x^k\|} \to \bar{v}$ and $\|\bar{v}\| = 1$. By Lemma 1.1, we obtain $\bar{v} \in 0^+\mathcal{F}(p)$.

Since $f(x^k, p) \to -\infty$, we can assume that

$$f(x^k, p) = \frac{1}{2}(x^k)^T Q x^k + q^T x^k \leq 0 \quad \forall k \geq 1. \tag{1.20}$$

Multiplying the inequality in (1.20) by $\|x^k\|^{-2}$ and letting $k \to \infty$, one has $\bar{v}^T Q \bar{v} \leq 0$. From this and the assumption (1.17) it follows that $\bar{v}^T Q \bar{v} = 0$. Since $\bar{v} \in 0^+\mathcal{F}(p)$, we can deduce that

$$(Qx^k + q)^T \bar{v} \geq 0$$

by the assumption (1.18). By repeating the argument as in the proof of Theorem 1.1, we can find $\delta > 0$ and $k_2 > 0$ such that

$$x^k - t\bar{v} \in S_k \quad \text{and} \quad \|x^k - t\bar{v}\| < \|x^k\| \quad \forall k \geq k_2, \quad \forall t \in (0, \delta).$$

This contradicts the fact that x^k is the smallest norm element in S_k. Therefore, we conclude that $f^* > -\infty$, which proves the theorem. $\square$

The following example shows that (A_2) can not be dropped from the assumptions of Theorem 1.2.

Example 1.3. *Consider again the problem in Example 1.2. Both conditions* (1.17) *and* (1.18) *are satisfied because of*

$$v^T Q v = -4v_2 v_3 = 0;$$

and

$$(Qx + q)^T v = 2v_1 - 2x_3 v_2 - 2x_2 v_3 = 2v_1 \geq 0$$

for every $x = (x_1, x_2, x_3) \in \mathcal{F}(p)$ *and for every* $v = (v_1, v_2, v_3) \in 0^+\mathcal{F}(p)$. *Condition* (A_2) *is not satisfied and this problem has no solution.*

To illustrate for Theorem 1.2, we consider the following example.

Example 1.4. *Let us consider the problem* $(QP(p))$ *with* $m = 2$, $n = 3$, *and*

$$Q = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad q = \begin{pmatrix} 0 \\ 2 \\ -5 \end{pmatrix}, \qquad Q_1 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$q_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad c_1 = 0, \qquad Q_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad q_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad c_2 = 0.$$

This problem can be rewritten as follows

$$\min \left\{ f(x,p) = \frac{1}{2}x_3^2 + 2x_2 - 5x_3 : x \in \mathcal{F}(p) \right\},$$

where $\mathcal{F}(p) = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2 \le 0; x_2^2 + x_3 \le 0\}$.

It can be verified that $\mathcal{F}(p) \neq \emptyset$ *and*

$$0^+\mathcal{F}(p) = \{(v_1, v_2, v_3) \in \mathbb{R}^3 : v_1 = 0, v_2 = 0, v_3 \le 0\}.$$

For every $v = (v_1, v_2, v_3) \in 0^+\mathcal{F}(p)$, *we obtain* $v^T Q v = v_3^2 \ge 0$. *Thus* (1.17) *is satisfied. Let any* $v = (v_1, v_2, v_3) \in 0^+\mathcal{F}(p)$ *such that* $v^T Q v = 0$, *we then get* $v = (0, 0, 0)$. *Hence* (1.18) *is satisfied. Furthermore, it is easy to check that* (A_2) *holds. According to Theorem 1.2, this problem has a solution.*

1.4. Conclusions

In this chapter, we have established sufficient conditions to solution existence of the nonconvex QCQP problem through a Frank-Wolfe type theorem (Theorem 1.1) and an Eaves type theorem (Theorem 1.2). The continuity of the global optimal solution map will be studied in the next chapter.

Chapter 2

Stability for global, local and stationary solution sets

In this chapter, we characterize the continuity of the global, local and stationary solution maps. In Section 2.1, we establish conditions for the continuity of the global optimal solution map by using results in Chapter 1. Section 2.2 gives a necessary and sufficient condition for the lower semicontinuity of the local optimal solution map. The upper semicontinuity of the stationary solution map is presented in Section 2.3. A stability result for stationary solution set is investigated in the connection with parametric extended affine variational inequalities.

The material of this chapter is taken from [77, 79, 102].

2.1. Continuity of the global optimal solution map

Using the obtained results on solution existence in Chapter 1, this section characterizes continuity of the global optimal solution map of QCQP problems. First of all, we present the following assumptions and auxiliary results.

2.1.1. Assumptions and auxiliary results

We recall the notion of upper semicontinuity and lower semicontinuity of multifunctions.

A multifunction $F : X \subset \mathbb{R}^s \rightrightarrows \mathbb{R}^n$ is said to be *upper semicontinuous* at $\bar{p} \in \mathbb{R}^s$ if for each open set V containing $F(\bar{p})$ there exists $\delta > 0$ such that $F(p) \subset V$ for every $p \in \mathbb{R}^s$ satisfying $\|p - \bar{p}\| < \delta$.

A multifunction $F : X \subset \mathbb{R}^s \rightrightarrows \mathbb{R}^n$ is said to be *lower semicontinuous* at $\bar{p} \in \mathbb{R}^n$ if $F(\bar{p}) \neq \emptyset$ and, for each open set V satisfying $F(\bar{p}) \cap V \neq \emptyset$, there exists $\delta > 0$ such that $F(p) \cap V \neq \emptyset$ for every $p \in \mathbb{R}^s$ satisfying $\|p - \bar{p}\| < \delta$.

The notion of upper semicontinuity stated herein is quite standard (see, for instance, [109, p. 451]). It is very different from the concept of stability considered by Gowda and Pang [40, Def. 1]. The notion of lower semicontinuity stated herein agrees with that considered in [109, p. 451], but differs slightly from the one given in [4, p. 39].

Studying the continuity of the solution mapping plays a central role in stability theory in mathematical programming and variational inequality. For example, the continuity analysis can help us to make convergent analysis of the algorithm. The problems concern on how the solution mappings vary with the changes in the parameters. Stability requires that for a small perturbations or a small measurement error of the parameters, the induced perturbations of the solution mapping is also very small. In recent years, there have been many results on the (semi)continuity, especially the lower semicontinuity of the solution mapping for parametric nonlinear programming problems, variational inequalities, and equilibrium problems in the literature (see, for instance, [1–3, 18, 40, 41, 49, 50, 56, 60, 101, 108]).

We say that the system

$$g_i(x, p) \leq 0, \ i = 1, \dots, m,$$

satisfies *Slater Constraint Qualification (SCQ)* if there exists $x^0 \in \mathbb{R}^n$ such that $g_i(x^0, p) < 0$ for all $i = 1, \ldots, m$. We also say that $(QP(p))$ satisfies (SCQ) if the system $g_i(x, p) \leq 0,\ i = 1, \ldots, m$, satisfies (SCQ).

The following result is well-known (see, for example, [10, Theorem 3.1.5] and [87, Theorem 1]).

Lemma 2.1. *The system $g_i(x, p) \leq 0,\ i = 1, \ldots, m$, satisfies (SCQ) if and only if the set-valued map $\mathcal{F} : \mathbb{P} \rightrightarrows \mathbb{R}^n$, defined by*

$$\mathcal{F}(\tilde{p}) = \{x \in \mathbb{R}^n : \ g_i(x, \tilde{p}) \leq 0,\ i = 1, \ldots, m\} \quad \forall \tilde{p} \in \mathbb{P},$$

is lower semicontinuous at p.

Remark 2.1. *If the system $g_i(x, p) \leq 0,\ i = 1, \ldots, m$, does not satisfy (SCQ), then there exists $p^k \to p$ such that, for each k, the system $g_i(x, p^k) \leq 0,\ i = 1, \ldots, m$, has no solution. Indeed, suppose that the system $g_i(x, p) \leq 0,\ i = 1, \ldots, m$, does not satisfy (SCQ). Let*

$$p^k = (Q_1, q_1, c_1^k, \ldots, Q_m, q_m, c_m^k) \to p$$

satisfying $c_i^k > c_i$, for every $i = 1, \ldots, m$. Then, for any $\bar{x} \in \mathbb{R}^n$, there exists $i_{\bar{x}} \in \{1, \ldots, m\}$ such that $g_{i_{\bar{x}}}(\bar{x}, p) \geq 0$. We have

$$g_{i_{\bar{x}}}(\bar{x}, p^k) > g_{i_{\bar{x}}}(\bar{x}, p) \geq 0.$$

This implies $g_{i_{\bar{x}}}(\bar{x}, p^k) > 0$. Then, the system $g_i(x, p^k) \leq 0,\ i = 1, \ldots, m$, has no solution for every k.

The Mangasarian-Fromovitz Constraint Qualification (MFCQ) is satisfied at $\bar{x} \in \mathcal{F}(p)$ if there exists $v^0 \in \mathbb{R}^n$ such that

$$(Q_i\bar{x} + q_i)^T v^0 < 0 \ \ \forall i \in I(\bar{x}, p),$$

where $I(\bar{x}, p) := \{i \in \{1, \ldots m\} : g_i(\bar{x}, p) = 0\}$ is the *active constraint index* set.

Remark 2.2. *Since Q_i, $i = 1, \ldots, m$, are positive semidefinite, (SCQ) is equivalent to $(MFCQ)$ (see, for instance, [65, p.47-48]).*

An important assumption used in our proof is given below.

Assumption $(\mathbf{A}_3)$ *The set* $\mathcal{F}(p) \neq \emptyset$ *and* $v^TQv > 0$ *for every nonzero vector* $v \in 0^+\mathcal{F}(p)$.

In [36, Theorem 3.3], Gauvin and Dubeau proposed the assumption of the uniform compactness of $\mathcal{F}(\tilde{p})$ near p (i.e. there is a neighborhood $N(p)$ of p such that the closure of $\cup_{\tilde{p}\in N(p)}\mathcal{F}(\tilde{p})$ is compact). Clearly, (A_3) holds if $\mathcal{F}(p)$ is nonempty and bounded. Thus (A_3) is weaker than the uniform compactness of $\mathcal{F}(\tilde{p})$ near p applied for $(QP(p))$.

Denote by S the set of all $p \in \mathbb{P}$ such that $(QP(p))$ satisfies (A_3). We have the following lemma.

Lemma 2.2. S *is open in* $\mathbb{P}$.

Proof. Suppose that, contrary to our claim, S is not open in $\mathbb{P}$. Then, there exists $\{p^k\} \subset \mathbb{P} \setminus S$ converging to $p \in S$. For each p^k, there exists $v^k \in \mathbb{R}^n$ such that

$$\|v^k\| = 1, Q_i^k(v^k) = 0, (q_i^k)^T v^k \leq 0,\ i = 1, \dots, m \ \text{ and } \ (v^k)^T Q^k v^k \leq 0. \tag{2.1}$$

Without loss of generality, we may assume that the sequence $\{v^k\}$ itself converges to $\hat{v}$ for some $\hat{v} \in \mathbb{R}^n$. Taking the limits in (2.1) as $k \to \infty$ yields

$$\|\hat{v}\| = 1, Q_i\hat{v} = 0, q_i^T\hat{v} \leq 0,\ i = 1, \dots, m \ \text{ and } \ \hat{v}^T Q\hat{v} \leq 0.$$

This contradicts the fact that $p \in S$, which completes the proof. $\square$

2.1.2. Upper semicontinuity of the global optimal solution map

The upper semicontinuity of the global optimal solution map $G(\cdot)$ is characterized as follows.

Theorem 2.1. *Consider the problem* $(QP(p))$. *Assume that* (SCQ) *and* (A_3) *hold at* $\bar{p} = (\bar{Q}, \bar{q}, \bar{Q}_1, \bar{q}_1, \bar{c}_1, \ldots, \bar{Q}_m, \bar{q}_m, \bar{c}_m) \in \mathbb{P}$. *Then,* $G(\cdot)$ *is upper semicontinuous at* $\bar{p}$.

Proof. To obtain a contradiction, suppose that $G(\bar{p}) \neq \emptyset$ and there exist an open set U containing $G(\bar{p})$, a sequence $\{p^k\}$ converging to $\bar{p}$ and a sequence $\{x^k\}$ such that $x^k \in G(p^k) \setminus U$ for every $k \in \mathbb{N}$.

If $\{x^k\}$ is bounded, then there is no loss of generality in assuming that $x^k \to \hat{x}$ for some $\hat{x} \in \mathbb{R}^n$. It is clear that $\hat{x} \in \mathcal{F}(\bar{p})$. Fix any $x \in \mathcal{F}(\bar{p})$. By the assumption that (SCQ) holds at $\bar{p}$ and by Lemma 2.1, $\mathcal{F}(\cdot)$ is lower semicontinuous at $\bar{p}$. Hence there exists a sequence $\{\xi^k\}$, $\xi^k \in \mathcal{F}(p^k)$ for all $k \in N$, such that $\lim\limits_{k\to\infty} \xi^k = x$. Since $x^k \in G(p^k)$, we have $f(x^k, p^k) \leq f(\xi^k, p^k)$. Letting $k \to \infty$ we get $f(\hat{x}, \bar{p}) \leq f(x, \bar{p})$. This shows $\hat{x} \in G(\bar{p}) \subset U$. We have arrived at a contradiction, because $x^k \notin U$ for all k and U is open. Hence $\{x^k\}$ is unbounded.

Let us assume, without loss of generality, that $\|x^k\|^{-1}x^k \to \bar{v}$ and $\|\bar{v}\| = 1$. By Lemma 1.1, we have $\bar{v} \in 0^+\mathcal{F}(\bar{p})$. Fix any $y \in \mathcal{F}(\bar{p})$. By the assumption that (SCQ) holds at $\bar{p}$, there exists a sequence $\{y^k\}$, $y^k \in \mathcal{F}(p^k)$ for all k and $y^k \to y$. Dividing the inequality

$$\frac{1}{2}(x^k)^T Q^k x^k + (q^k)^T x^k \leq \frac{1}{2}(y^k)^T Q^k y^k + (q^k)^T y^k$$

by $\|x^k\|^2$ and letting $k \to \infty$ yields $\bar{v}^T \bar{Q} \bar{v} \leq 0$, contrary to (A_3). This completes the proof. □

2.1.3. Lower semicontinuity of the global optimal solution map

The following theorem shows the necessary and sufficient condition for the lower semicontinuity of the global optimal solution map $G(\cdot)$.

Theorem 2.2. *Consider the problem* $(QP(p))$. *The map* $G(\cdot)$ *is lower* *semicontinuous at* $\bar{p} = (\bar{Q}, \bar{q}, \bar{Q}_1, \bar{q}_1, \bar{c}_1, \ldots, \bar{Q}_m, \bar{q}_m, \bar{c}_m) \in \mathbb{P}$ *if and only if* (SCQ) *and* (A_3) *hold at* $\bar{p}$ *and* $G(\bar{p})$ *is a singleton.*

Proof. Necessity: On the contrary, suppose that $G(\cdot)$ is lower semicontinuous at $\bar{p}$ but $G(\bar{p})$ is not a singleton. From $G(\bar{p}) \neq \emptyset$, there exist $\bar{x}, \bar{y} \in G(\bar{p})$ such that $\bar{x} \neq \bar{y}$. Choose $\hat{q} \in \mathbb{R}^n$ such that

$$\|\hat{q}\| = 1, \ \hat{q}^T\bar{x} > \hat{q}^T\bar{y}.$$

Clearly, there exists an open neighborhood U of $\bar{x}$ such that

$$\hat{q}^T x > \hat{q}^T\bar{y} \ \ \forall x \in U. \tag{2.2}$$

Given any $\delta > 0$, we fix a number $\varepsilon \in (0, \delta)$ and put $q^\varepsilon = \bar{q} + \varepsilon\hat{q}$ and $p^\varepsilon = (\bar{Q}, q^\varepsilon, \bar{Q}_1, \bar{q}_1, \bar{c}_1, \ldots, \bar{Q}_m, \bar{q}_m, \bar{c}_m)$. Then, $\|p^\varepsilon - \bar{p}\| = \|q^\varepsilon - \bar{q}\| = \varepsilon < \delta$. Our next goal is to show that $G(p^\varepsilon) \cap U = \emptyset$. For any $x \in \mathcal{F}(\bar{p}) \cap U$, since $\bar{x}, \bar{y} \in G(\bar{p})$, by (2.2) we have

$$\frac{1}{2}x^T\bar{Q}x + (q^\varepsilon)^T x > \frac{1}{2}\bar{x}^T\bar{Q}\bar{x} + \bar{q}^T\bar{x} + \varepsilon\hat{q}^T\bar{y} = \frac{1}{2}\bar{y}^T\bar{Q}\bar{y} + (q^\varepsilon)^T\bar{y}.$$

It follows $x \notin G(p^\varepsilon)$. Hence for the chosen neighborhood U of $\bar{x} \in G(\bar{p})$ and for every $\delta > 0$, there exists $q^\varepsilon \in \mathbb{R}^n$ such that $\|q^\varepsilon - \bar{q}\| < \delta$ and $G(p^\varepsilon) \cap U = \emptyset$. This contradicts the fact that $G(\cdot)$ is lower semicontinuous at $\bar{p}$. Therefore $G(\bar{p})$ is a singleton.

Suppose that $G(\cdot)$ is lower semicontinuous at $\bar{p}$ but (SCQ) is not satisfied at $\bar{p}$. Then, by Remark 2.1, there exists a sequence $\{p^k\} \subset \mathbb{P}$ converging to $\bar{p}$ such that $\mathcal{F}(p^k) = \emptyset$ for every k. Therefore $G(p^k) = \emptyset$ for every k. This contradicts the fact that $G(\cdot)$ is lower semicontinuous at $\bar{p}$.

Suppose that $G(\cdot)$ is lower semicontinuous at $\bar{p}$ but (A_3) is not satisfied at $\bar{p}$. There exists a nonzero vector $\bar{v} \in 0^+\mathcal{F}(\bar{p})$ such that $\bar{v}^T\bar{Q}\bar{v} \leq 0$. Then, $\mathcal{F}(\bar{p})$ is unbounded. For every $\delta > 0$, we obtain that $\bar{v}^T(\bar{Q} - \delta I)\bar{v} < 0$. Let $p^\delta := (\bar{Q} - \delta I, \bar{q}, \bar{Q}_1, \bar{q}_1, \bar{c}_1, \ldots, \bar{Q}_m, \bar{q}_m, \bar{c}_m)$ converging to $\bar{p}$. For any $x \in \mathcal{F}(\bar{p})$, one has

$$f(x + t\bar{v}, p^\delta) = \frac{1}{2}(x + t\bar{v})^T(\bar{Q} - \delta I)(x + t\bar{v}) + \bar{q}^T(x + t\bar{v}) \to -\infty$$

as $t \to +\infty$. Hence $G(p^\delta) = \emptyset$, contrary to the fact that $G(\cdot)$ is lower semicontinuous at $\bar{p}$.

Sufficiency: Suppose that (SCQ) and (A_3) hold at $\bar{p}$ and $G(\bar{p})$ is a singleton. Let U be an open set containing the unique solution $\bar{x} \in G(\bar{p})$. By the assumption that (SCQ) holds at $\bar{p}$, there exists $\delta_1 > 0$ such that $\mathcal{F}(\tilde{p}) \neq \emptyset$ for every pair $\tilde{p}$ satisfying $\|\tilde{p} - \bar{p}\| < \delta_1$ (see Lemma 2.1). By (A_3) and by Lemma 2.2, there exists $\delta_2 > 0$ such that $x^T\widetilde{Q}x$ is positive definite on the cone $0^+\mathcal{F}(\tilde{p})$ for every $\tilde{p}$ satisfying $\|\tilde{p} - \bar{p}\| < \delta_2$. Let $\delta := \min\{\delta_1, \delta_2\}$. By Corollary 1.3, we have $G(\tilde{p}) \neq \emptyset$ for every $\tilde{p}$ satisfying $\|\tilde{p} - \bar{p}\| < \delta$.

Since (SCQ) and (A_3) hold at $\bar{p}$, it follows from Theorem 2.1 that $G(\cdot)$ is upper semicontinuous at $\bar{p}$. Hence $G(\tilde{p}) \subset U$ for every $\tilde{p}$ satisfying $\|\tilde{p} - \bar{p}\| < \delta$ if $\delta > 0$ is small enough. For such δ, from what has been said it follows that $G(\tilde{p}) \cap U \neq \emptyset$ for every $\tilde{p}$ satisfying $\|\tilde{p} - \bar{p}\| < \delta$. This shows that $G(\cdot)$ is lower semicontinuous at $\bar{p}$. This ends the proof. $\square$

Example 2.1. *We consider the problem $(QP(p))$ with $n = 2$ and $m = 1$. Let $\bar{p} = (\bar{Q}, \bar{q}, \bar{Q}_1, \bar{q}_1, \bar{c}_1) \in \mathbb{P}$, where*

$$\bar{Q} = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}, \bar{q} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \bar{Q}_1 = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \quad \bar{q}_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \bar{c}_1 = -1.$$

This problem can be rewritten as follows

$$\min \left\{ f(x, \bar{p}) = -\frac{1}{2}x_2^2 + x_1, x \in \mathcal{F}(\bar{p}) \right\},$$

where $\mathcal{F}(\bar{p}) = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 \leq 1\}$. Clearly, (SCQ) and (A_3) hold at $\bar{p}$. It is easy to check that $G(\bar{p}) = \{\bar{x} = (-1, 0)\}$. By Theorems 2.1-2.2, the global optimal solution map $G(\cdot)$ is continuous at $\bar{p}$.

Remark 2.3. *Gauvin and Dubeau [36, Lemma 3.2] prove that the global optimal solution map of the mathematical programming problem is upper semicontinuous at a parametric $\bar{y}$ if it is uniformly compact near $\bar{y}$ and the global optimal solution set is nonempty. Clearly, (A_3) is weaker than the uniform compactness of $\mathcal{F}(p)$ near $\bar{p}$ [36] applied for $(QP(p))$.*

2.2. Semicontinuity of the local optimal solution map

In this section, we propose a necessary and sufficient condition for the lower semicontinuity of the local optimal solution map $L(\cdot)$. The *isolated local optimal solution set* of $(QP(p))$ will be denoted by $IL(p)$. The main result is presented below.

Theorem 2.3. *Consider the problem* $(QP(p))$*. The map* $L(\cdot)$ *is lower semicontinuous at* $\bar{p} = (\bar{Q}, \bar{q}, \bar{Q}_1, \bar{q}_1, \bar{c}_1, \dots, \bar{Q}_m, \bar{q}_m, \bar{c}_m) \in \mathbb{P}$ *if and only if* $(QP(\bar{p}))$ *satisfies (SCQ) and the set of local optimal solutions coincides with the set of isolated local optimal solutions, i.e.,* $L(\bar{p}) = IL(\bar{p})$.

Proof. Necessity: Suppose that $L(\cdot)$ is lower semicontinuous at $\bar{p}$ but $(QP(\bar{p}))$ does not satisfy (SCQ). According to Remark 2.1, there exists a sequence $p^k \in \Omega$ converging to $\bar{p}$ such that $\mathcal{F}(p^k) = \emptyset$ for every k. This implies $L(p^k) = \emptyset$ for every k, contrary to the assumption that $L(\cdot)$ is lower semicontinuous at $\bar{p}$.

Suppose that $L(\bar{p}) \neq IL(\bar{p})$. Since $IL(\bar{p}) \subset L(\bar{p})$, there exists $\bar{x} \in L(\bar{p}) \setminus IL(\bar{p})$. Then, there are an open set U containing $\bar{x}$ such that $f(z, \bar{p}) \geq f(\bar{x}, \bar{p})$ for every $z \in \mathcal{F}(\bar{p}) \cap U$ and a local optimal solution $\bar{y} \in L(\bar{p}) \cap U$ such that $\bar{x} \neq \bar{y}$. We can choose $\tilde{q} \in \mathbb{R}^n$ such that $\tilde{q}^T \bar{x} > \tilde{q}^T \bar{y}$ and $\|\tilde{q}\| = 1$. There exists a neighborhood V of $\bar{x}$ such that

$$\tilde{q}^T x > \tilde{q}^T \bar{y} \ \ \forall x \in V. \tag{2.3}$$

Let $W := U \cap V$. For given $\delta > 0$ and for $t \in (0, \delta)$, let

$$p^t = (\bar{Q}, q^t, \bar{Q}_1, \bar{q}_1, \bar{c}_1, \dots, \bar{Q}_m, \bar{q}_m, \bar{c}_m),$$

with $q^t = \bar{q} + t\tilde{q}$. Then, $\|p^t - \bar{p}\| = \|q^t - \bar{q}\| = t < \delta$. For every

$z \in \mathcal{F}(\bar{p}) \cap W$, from (2.3), we obtain that

$$\begin{aligned}\tfrac{1}{2}z^T\bar{Q}z + (q^t)^T z &= \tfrac{1}{2}z^T\bar{Q}z + \bar{q}^T z + t\tilde{q}^T z \geq \tfrac{1}{2}\bar{x}^T\bar{Q}\bar{x} + \bar{q}^T\bar{x} + t\tilde{q}^T z \\ &> \tfrac{1}{2}\bar{x}^T\bar{Q}\bar{x} + \bar{q}^T\bar{x} + t\tilde{q}^T\bar{y} = \tfrac{1}{2}\bar{y}^T\bar{Q}\bar{y} + (q^t)^T\bar{y}.\end{aligned}$$

Hence $z \notin L(p^t)$. This leads to $L(p^t) \cap W = \emptyset$. Hence for the chosen neighborhood W of $\bar{x} \in L(\bar{p})$ and for every $\delta > 0$, there exists $q^t \in \mathbb{R}^n$ satisfying $\|q^t - \bar{q}\| < \delta$ such that $L(p^t) \cap W = \emptyset$. This contradicts the assumption that $L(\cdot)$ is lower semicontinuous at $\bar{p}$. Thus, $L(\bar{p}) = IL(\bar{p})$.

Sufficiency: Suppose that the problem $(QP(\bar{p}))$ satisfies (SCQ) and $L(\bar{p}) = IL(\bar{p})$. Take any open set $V \subset \mathbb{R}^n$ such that $L(\bar{p}) \cap V \neq \emptyset$. Fix $\bar{x} \in V \cap L(\bar{p})$. Since $L(\bar{p}) = IL(\bar{p})$, $\bar{x} \in V \cap IL(\bar{p})$. Hence there exists an open ball $B(\bar{x}, \epsilon) \subset V$ such that $f(x, \bar{p}) > f(\bar{x}, \bar{p})$ for every $x \in (\mathcal{F}(\bar{p}) \cap B(\bar{x}, \epsilon)) \setminus \{\bar{x}\}$. It follows that $\bar{x}$ is the unique global optimal solution of the following auxiliary problem

$$\min\{f(x, p) : x \in \mathcal{F}(p) \cap \bar{B}(\bar{x}, \epsilon/2)\}, \tag{AP}$$

where $\bar{B}(\bar{x}, \epsilon/2)$ is the closure of $B(\bar{x}, \epsilon/2)$.

Next, we show that the global optimal solution map $G_{AP}(\cdot)$ of the problem (AP) is upper semicontinuous at $\bar{p}$. Indeed, suppose that $G_{AP}(\cdot)$ is not upper semicontinuous at $\bar{p}$, that is, there exist an open set W containing $G_{AP}(\bar{p})$, a sequence $\{p^k\}$ converging to $\bar{p}$ and a sequence $\{y^k\}$ satisfying $y^k \in G_{AP}(p^k) \setminus W$ for every $k \in \mathbb{N}$. Since $\{y^k\}$ is bounded, without loss of generality, we may assume that $y^k \to \bar{y}$ for some $\bar{y} \in \mathbb{R}^n$. From $y^k \in G_{AP}(p^k)$ it follows that $y^k \in \mathcal{F}(p^k) \cap \bar{B}(\bar{x}, \epsilon/2)$. Letting $k \to \infty$, we have $\bar{y} \in \mathcal{F}(\bar{p}) \cap \bar{B}(\bar{x}, \epsilon/2)$. Since $(ET(\bar{p}))$ satisfies (SCQ), $\mathcal{F}(\cdot)$ is lower semicontinuous at $\bar{p}$ (see Lemma 2.1). Hence for $\bar{x} \in \mathcal{F}(\bar{p}) \cap B(\bar{x}, \epsilon/2)$, there exists x^k converging to $\bar{x}$ such that $x^k \in \mathcal{F}(p^k) \cap B(\bar{x}, \epsilon/2)$ for all $k \in \mathbb{N}$. From $y^k \in \mathrm{G}_{AP}(p^k)$ it follows $f(y^k, p^k) \leq f(x^k, p^k)$. Letting $k \to \infty$ yields $f(\bar{y}, \bar{p}) \leq f(\bar{x}, \bar{p})$. Since $\bar{x}$ is the unique global optimal solution of (P), we have $\bar{y} = \bar{x}$. Hence $\bar{y} \in G_{AP}(\bar{p}) \subset W$.

This contradicts the fact that W is open and $y^k \notin W$ for all k. Thus $G_{AP}(\cdot)$ is upper semicontinuous at $\bar{p}$

Take any open set U such that $G_{AP}(\bar{p}) \cap U \neq \emptyset$. Since $\mathcal{F}(\cdot)$ is lower semicontinuous and $\mathcal{F}(\bar{p}) \cap B(\bar{x}, \epsilon/2) \neq \emptyset$, there exists $\delta_1 > 0$ such that $\mathcal{F}(\tilde{p}) \cap B(\bar{x}, \epsilon/2) \neq \emptyset$ for every $\tilde{p}$ satisfying $\|\tilde{p} - \bar{p}\| < \delta_1$. Hence for such a number δ_1, we have $G_{AP}(\tilde{p}) \neq \emptyset$. Since $G_{AP}(\cdot)$ is upper semicontinuous at $\bar{p}$, $G_{AP}(\tilde{p}) \subset U$ for every $\tilde{p}$ satisfying $\|\tilde{p} - \bar{p}\| < \delta_1$ for $\delta_1 > 0$ small enough. This follows that $G_{AP}(\cdot)$ is lower semicontinuous at $\bar{p}$.

Since $G_{AP}(\bar{p}) \cap\ B(\bar{x}, \epsilon/2) \neq \emptyset$, there exists a number $\delta_2 > 0$ such that $G_{AP}(\tilde{p}) \cap B(\bar{x}, \epsilon/2) \neq \emptyset$ for every $\tilde{p} \in \Omega$ satisfying $\|\tilde{p} - \bar{p}\| < \delta_2$. This leads to $L(\tilde{p}) \cap V \neq \emptyset$ for every $\tilde{p} \in \Omega$ satisfying $\|\tilde{p} - \bar{p}\| < \delta_2$. Therefore $L(\cdot)$ is lower semicontinuous at $\bar{p}$. The theorem is proved. $\square$

The next example tells us that the local optimal solution map $L(\cdot)$ is not lower semicontinuous if there exists a local optimal solution which is not an isolated local optimal solution.

Example 2.2. *Consider $(QP(\bar{p}))$ with $n = 2, m = 1$,*

$$\bar{Q} = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}, \quad \bar{q} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \bar{Q}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \quad \bar{q}_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \ \bar{c}_1 = -2.$$

This problem can be rewritten as follows

$$\min\left\{ f(x, \bar{p}) = \frac{1}{2}(-x_1^2 - 2x_2^2) : x \in \mathcal{F}(\bar{p}) \right\},$$

where $\mathcal{F}(\bar{p}) = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + 2x_2^2 \leq 2\}$.

Let $T := \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + 2x_2^2 = 2\}$. We obtain that

$$f(x, \bar{p}) = \frac{1}{2}(-x_1^2 - 2x_2^2) \geq \frac{1}{2}(-2) = -1,$$

and $f(x, \bar{p}) = -1$ for all $x \in T$. Hence $\emptyset \neq T \subseteq G(\bar{p}) \subseteq L(\bar{p})$ and $T \nsubseteq IL(\bar{p})$. This implies $L(\bar{p}) \neq IL(\bar{p})$. From Theorem 2.3 it follows that $L(\cdot)$ is not lower semicontinuous at $\bar{p}$.

2.3. Stability of stationary solutions

In this section, the upper semicontinuity of the stationary solution map is characterized. A stability result for stationary solution set is also investigated in connection with parametric extended affine variational inequalities.

2.3.1. Preliminaries

Recall that x is a *stationary solution* of the problem $(QP(p))$ if there exists Lagrange multiplier $\lambda \in \mathbb{R}^m$ satisfying the following *Karush-Kuhn-Tucker (KKT) condition*:

$$Qx + q + \sum_{i=1}^{m} \lambda_i (Q_i x + q_i) = 0, \tag{2.4}$$

$$\lambda \geq 0, \ g_i(x,p) \leq 0, \tag{2.5}$$

$$\lambda_i g_i(x,p) = 0, \ i = 1, \dots, m. \tag{2.6}$$

The pair (x, λ) satisfying (2.4)–(2.6) is called a *KKT pair* of the problem $(QP(p))$ (see, for instance, [23, 54]).

The stationary solution set of $(QP(p))$ is denoted by $S(p)$. It is well-known that (see [18]), under (SCQ),

$$G(p) \subset L(p) \subset S(p) \subset \mathcal{F}(p).$$

We have the following lemma.

Lemma 2.3. (see [35]) *If the problem $(QP(p))$ satisfies (SCQ), then, for each $x \in \mathcal{F}(p)$, the set of multipliers corresponding to x is bounded.*

2.3.2. Upper semicontinuity of the stationary solution map

Denote

$$Null(Q) := \{x \in \mathbb{R}^n : Qx = 0\}.$$

The following result gives a sufficient condition for the upper semicontinuity of the stationary solution map $S(\cdot)$.

Theorem 2.4. *For $\bar{p} = (\bar{Q}, \bar{q}, \bar{Q}_1, \bar{q}_1, \bar{c}_1, \ldots, \bar{Q}_m, \bar{q}_m, \bar{c}_m) \in \mathbb{P}$, if $(QP(\bar{p}))$ satisfies (SCQ) and*

$$Null(\bar{Q}) \cap 0^+\mathcal{F}(\bar{p}) = \{0\}, \tag{2.7}$$

then $S(\cdot)$ is upper semicontinuous at $\bar{p}$.

Proof. Suppose the contrary that $S(\cdot)$ is not upper semicontinuous at $\bar{p}$. Then, there exist an open set V containing $S(\bar{p})$, a sequence $\{p^k\} \subset \mathbb{P}$ converging to $\bar{p}$ and a sequence $\{x^k\}$ with $x^k \in S(p^k)$ such that $x^k \notin V$. Since $(QP(p))$ satisfies (SCQ) at $\bar{p}$, it also satisfies (SCQ) at p^k for k large enough. Since $x^k \in S(Q^k, q^k, p^k)$, there exists $\lambda^k \in \mathbb{R}^m$ satisfying:

$$Q^k x^k + q^k + \sum_{i=1}^{m} \lambda_i^k (Q_i^k x^k + q_i^k) = 0, \tag{2.8}$$

$$\lambda^k \geq 0, \ g_i(x^k, p^k) \leq 0, \tag{2.9}$$

$$\lambda_i^k g_i(x^k, p^k) = 0, i = 1, \ldots, m, \tag{2.10}$$

and $\{\lambda^k\}$ is bounded. Without loss of generality, we can assume that λ^k converges to $\bar{\lambda}$ for some $\bar{\lambda} \in \mathbb{R}^m$. We consider the following two cases:

Case 1: $\{x^k\}$ is bounded. Then, there exists a sequence $\{k_i\} \subset \{k\}$ such that $x^{k_i} \to \bar{x}$ for some $\bar{x} \in \mathbb{R}^n$. Passing (2.8)–(2.10) to the limits as $i \to \infty$, we deduce that

$$\bar{Q}\bar{x} + \bar{q} + \sum_{i=1}^{m} \bar{\lambda}_i (\bar{Q}_i \bar{x} + \bar{q}_i) = 0,$$

$$\bar{\lambda} \geq 0, \ g_i(\bar{x}, \bar{p}) \leq 0,$$

$$\bar{\lambda}_i g_i(\bar{x}, \bar{p}) = 0, i = 1, \ldots, m.$$

These give $\bar{x} \in S(\bar{p}) \subset V$, contrary to the fact that $x^{k_i} \notin V$ and V is open.

Case 2: $\{x^k\}$ is unbounded. Then, $\|x^k\| > 0$ for k large enough. Without loss of generality, we may assume that $x^k/\|x^k\| \to \hat{x}$ with $\|\hat{x}\| = 1$. From the second inequality of (2.9) and Lemma 1.1 it follows $\hat{x} \in 0^+\mathcal{F}(\bar{p})$. Dividing both sides of (4.29) by $\|x^k\|$ and letting $k \to \infty$ yields

$$\bar{Q}\hat{x} + \sum_{i=1}^{m} \bar{\lambda}_i \bar{Q}_i \hat{x} = 0.$$

Combining this with $\hat{x} \in 0^+\mathcal{F}(\bar{p})$ gives $\bar{Q}\hat{x} = 0$, that is, $\hat{x} \in Null(\bar{Q})$. Then, we obtain $\hat{x} \in Null(\bar{Q}) \cap 0^+\mathcal{F}(\bar{p})$, contrary to the assumption $Null(\bar{Q}) \cap 0^+\mathcal{F}(\bar{p}) = \{0\}$. Therefore $S(\cdot)$ is upper semicontinuous at $\bar{p}$. □

Remark 2.4. *According to Theorem 2.4, the assumption that (SCQ) is also a sufficient condition for the upper semicontinuity of the stationary solution map $S(\cdot)$ in the case where any component of p is perturbed. But the reverse, in general, is not true. This is shown in the following example.*

Example 2.3. *Consider the problem $(QP(p))$ with $n = 1, m = 2$, $\bar{Q} = 1, \bar{q} = 0$, $\bar{Q}_1 = 1, \bar{q}_1 = 0, \bar{r}_1 = -1/2, \bar{Q}_2 = 0, \bar{q}_2 = -1$, and $\bar{c}_2 = 1$. Since $\mathcal{F}(\bar{p}) = \{1\}$, (SCQ) is not satisfied at $\bar{p}$. For $t \in \mathbb{R}$, let $p^t = (\bar{Q}, \bar{q}, \bar{Q}_1, \bar{q}_1, \bar{c}_1, \bar{Q}_2, \bar{q}_2, \bar{c}_2 + t)$. We have*

$$S(p^t) = \begin{cases} \{1+t\} & \text{if } t \le 0, \\ \emptyset & \text{otherwise.} \end{cases}$$

Then, for each open set U containing $S(\bar{p}) = \{1\}$, there exists $\epsilon > 0$ such that $S(p^t) \subset U$ for every t satisfying $|t| < \epsilon$. Hence the multifunction $S(\bar{Q}, \bar{q}, \bar{Q}_1, \bar{q}_1, \bar{c}_1, \bar{Q}_2, \bar{q}_2, .)$ is upper semicontinuous at $\bar{c}_2$.

The following is an immediate consequence of Theorem 2.4.

Corollary 2.1. *For $\bar{p} = (\bar{Q}, \bar{q}, \bar{Q}_1, \bar{q}_1, \bar{c}_1, \ldots, \bar{Q}_m, \bar{q}_m, \bar{c}_m) \in \mathbb{P}$, if $(QP(\bar{p}))$ satisfies (SCQ) and one of the following conditions is satisfied:*

(i) $\mathcal{F}(\bar{p})$ *is bounded;*

(ii) $\bar{Q}$ *is nonsingular (that is,* $\det \bar{Q} \neq 0$*),*

then $S(\cdot)$ *is upper semicontinuous at* $\bar{p}$*.*

Proof. If one of the following conditions (i) and (ii) occurs then (2.7) is satisfied. The corollary follows from Theorem 2.4. □

2.3.3. A result on stability of stationary solutions

In this section, we presents a result on stability of the stationary solution set. We use the tools relate to *extended affine variational inequality (EAVI)* to prove the main result.

Let $S \subset \mathbb{R}^n$ be a closed convex set and F be a function on S. A *variational inequality (VI)* problem has the following form

$$\textit{Find } x \in S \textit{ such that } \langle F(x), y - x \rangle \geq 0 \ \ \forall y \in S. \qquad (VI(F,S))$$

VIs give a rather general and suitable format for many problems arising in economics, mathematical physics, and operations research.

Problem $(VI(F,S))$ reduces to the *affine variational inequality (AVI) problem* if S is a polyhedral convex set and $F(x) = Qx + q$ with Q being an $(n \times n)$-matrix and $q \in \mathbb{R}^n$. The stability of the AVI problems has been studied by many authors. Gowda and Pang [40] obtained several sufficient conditions for the boundedness and stability of solutions to the AVI problem. Robinson [88] studied the stability of the AVI problems by the nonemptiness and the boundedness of global optimal solution set for the case where Q is a positive semidefinite matrix. Some similar topics have been investigated by Gowda and Seidman [41]. Lee et al. [60] presented conditions for the upper and the lower semicontinuities of the solution map of AVI problems. Some Lipschitz continuous properties of the global optimal solution map of the AVI problem were discussed in [56, Chap. 7].

As $F(x) = Qx + q$ and S is an arbitrary closed convex set, the problem $(VI(F, S))$ reduces to the EAVI problem. Tam [101] presented some stability results for the EAVI problem. A survey on the parametric optimization problems and parametric variational inequalities was given by Yen [108].

In this section, we concern the EAVI problem as follows

$$\text{Find } x \in \mathcal{F}(p) \text{ such that } \langle Qx + q, y - x\rangle \geq 0 \ \ \forall y \in \mathcal{F}(p) \qquad (VI(p))$$

depending on the parameter $p \in \mathbb{P}$. The *solution set* of $VI(p)$ will be denoted by $SolVI(p)$. We have the following lemma.

Lemma 2.4. (See, for instance, [32, Proposition 1.3.4], [35]) *Consider the problem $(VI(p))$. Assume that the system $g_i(x, p), i = 1, \ldots, m$, satisfies (SCQ). Then, $x \in SolVI(p)$ if and only if there exists λ satisfying* (2.4)–(2.6). *Moreover, for each $x \in SolVI(p)$, the set of multipliers corresponding to x is bounded.*

Consequently, under the assumption (SCQ), from Lemma 2.4 it follows

$$S(p) = SolVI(p).$$

We use this important property to obtain several perturbation results for the problem $(QP(p))$ under some assumptions. Our results develop and complement the published ones in [101, Theorem 2.3], where the constraint set is convex and unperturbed. Among our proposed assumptions, there are some weaker than those used in the cited works. The main tools are the recession cone and Hartman-Stampacchia's Theorem (see [51, Theorem 3.1]).

The following theorem is the main result in this subsection.

Theorem 2.5. *For $\bar{p} = (\bar{Q}, \bar{q}, \bar{Q}_1, \bar{q}_1, \bar{c}_1, \ldots, \bar{Q}_m, \bar{q}_m, \bar{c}_m) \in \mathbb{P}$, assume that $(QP(\bar{p}))$ satisfies (SCQ) and the following two conditions are satisfied:*

(a_1) $\{h \in 0^+\mathcal{F}(\bar{p}) : h^T\bar{Q}h = 0\} \subset Null(\bar{Q})$;

(a_2) *If $\mathcal{F}(\bar{p})$ is unbounded then*

$$\limsup_{k\to\infty} \frac{(x^k)^T\bar{Q}x^k}{\|x^k\|^2} \geq 0$$

for every sequence $\{x^k\} \subset \mathcal{F}(\bar{p})$ satisfying $\|x^k\| \to \infty$.

Then, the following four assertions are equivalent:

(b_1) *There exists a number $\gamma > 0$ such that $S(\tilde{p})$ is nonempty for every $\tilde{p} \in \mathbb{P}$ satisfying $\|\tilde{p} - \bar{p}\| < \gamma$;*

(b_2) *$S(\bar{p})$ is nonempty and bounded;*

(b_3) $\left\{x \in \mathcal{F}(\bar{p}) : (\bar{Q}x + \bar{q})^T h > 0 \ \ \forall h \in 0^+\mathcal{F}(\bar{p}) \setminus \{0\}\right\} \neq \emptyset$;

(b_4) $\bar{q} \in int((0^+\mathcal{F}(\bar{p}))^* - \bar{Q}\mathcal{F}(\bar{p}))$,
where $(0^+\mathcal{F}(\bar{p}))^* = \{y \in \mathbb{R}^n : h^T y \geq 0 \ \ \forall h \in 0^+\mathcal{F}(\bar{p})\}$.

Proof. By a similar argument as in [56, Lemma 7.2], we obtain that $(b_3) \Leftrightarrow (b_4)$. We now show that $(b_1) \Rightarrow (b_2)$, $(b_2) \Rightarrow (b_3)$, and $(b_3) \Rightarrow (b_1)$.

$(b_1) \Rightarrow (b_2)$: Suppose that there exists a number $\gamma > 0$ such that $S(\tilde{p})$ is nonempty for every $\tilde{p} \in \mathbb{P}$ satisfying $\|\tilde{p} - \bar{p}\| < \gamma$. This implies $S(\bar{p}) \neq \emptyset$. To obtain a contradiction, suppose that $S(\bar{p})$ is unbounded, that is, there exists a sequence $\{y^k\} \subset S(\bar{p})$ such that $\|y^k\| \to \infty$. Without loss of the generality, we may assume that $\|y^k\|^{-1} y^k \to \bar{h}$ for some $\bar{h} \in \mathbb{R}^n \setminus \{0\}$. By Lemma 1.1, we obtain $\bar{h} \in 0^+\mathcal{F}(\bar{p})$. For any $z \in \mathcal{F}(\bar{p})$, for each k, by $y^k \in S(\bar{p})$, we have

$$\langle \bar{Q}y^k + \bar{q}, z - y^k \rangle \geq 0.$$

This follows
$$\langle \bar{Q}y^k + \bar{q}, z\rangle \geq \langle \bar{Q}y^k, y^k\rangle + \langle \bar{q}, y^k\rangle. \tag{2.11}$$
Dividing the both sides of (2.11) by $\|y^k\|^2$ and letting $k \to \infty$ gives $\langle \bar{Q}\bar{h}, \bar{h}\rangle \leq 0$. From the assumption (a_2) it follows $\langle \bar{Q}\bar{h}, \bar{h}\rangle \geq 0$. Hence $\langle \bar{Q}\bar{h}, \bar{h}\rangle = 0$. By the assumption (a_1), we have $\bar{h} \in Null(\bar{Q})$. Multiplying both sides of (2.11) by $\|y^k\|^{-1}$, taking limsup and using the assumption (a_2) yields
$$\langle \bar{Q}\bar{h}, z\rangle \geq \langle \bar{q}, \bar{h}\rangle. \tag{2.12}$$
From $h \in Null(\bar{Q})$ and (2.12),
$$\langle \bar{Q}z + \bar{q}, \bar{h}\rangle \leq \langle \bar{Q}z, \bar{h}\rangle + \langle \bar{Q}\bar{h}, z\rangle = 0. \tag{2.13}$$
Since z is chosen arbitrarily, (2.13) holds for every $z \in \mathcal{F}(\bar{p})$. Let
$$\tilde{p} := \big(\bar{Q}, \bar{q} - \frac{\gamma}{2}\bar{h}, \bar{Q}_1, \bar{q}_1, \bar{c}_1, \ldots, \bar{Q}_m, \bar{q}_m, \bar{c}_m\big).$$
We have
$$\|\tilde{p} - \bar{p}\| = \frac{\gamma}{2} < \gamma.$$
From (2.13) it follows that
$$\langle \tilde{Q}z + \tilde{q}, \bar{h}\rangle = \langle \bar{Q}z + \bar{q}, \bar{h}\rangle - \frac{\gamma}{2}\langle \bar{h}, \bar{h}\rangle < 0 \ \ \forall z \in \mathcal{F}(\tilde{p}) = \mathcal{F}(\bar{p}).$$
For each $z \in \mathcal{F}(\tilde{p})$, there exists $y = z + \bar{h} \in \mathcal{F}(\tilde{p})$ such that
$$\langle \tilde{Q}z + \tilde{q}, y - z\rangle = \langle \tilde{Q}z + \tilde{q}, \bar{h}\rangle < 0.$$
This implies $S(\tilde{p}) = \emptyset$, contrary to the assumption (b_1). Thus $S(\bar{p})$ is bounded.

$(b_2) \Rightarrow (b_3)$: Suppose that (b_2) holds, but (b_3) does not. Then, there exists $\bar{h} \in 0^+\mathcal{F}(\bar{p}) \setminus \{0\}$ such that
$$\langle \bar{Q}z + \bar{q}, \bar{h}\rangle \leq 0 \ \ \forall z \in \mathcal{F}(\bar{p}). \tag{2.14}$$
Since $S(\bar{p})$ is nonempty, there exists $\bar{x} \in S(\bar{p})$. Let $z^k := \bar{x} + k\bar{h}$ for $k = 1, 2, \ldots$ Since $\bar{h} \in 0^+\mathcal{F}(\bar{p})$, we have $z^k \in \mathcal{F}(\bar{p})$ for every k and $\|z^k\| \to +\infty$ as $k \to +\infty$. From (2.14),
$$\langle \bar{Q}z^k + \bar{q}, \bar{h}\rangle = \langle \bar{Q}\bar{x} + \bar{q}, \bar{h}\rangle + k\langle \bar{Q}\bar{h}, \bar{h}\rangle \leq 0 \ \ \forall k.$$

The latter inequality implies $\langle \bar{Q}\bar{h}, \bar{h}\rangle \leq 0$. On the other hand, we have

$$\frac{\langle \bar{Q}z^k, z^k\rangle}{\|z^k\|^2} = \frac{\langle \bar{Q}\bar{x}, \bar{x}\rangle}{\|\bar{x}+k\bar{h}\|^2} + \frac{k\langle \bar{Q}\bar{x}, \bar{h}\rangle}{\|\bar{x}+k\bar{h}\|^2} + \frac{k\langle \bar{Q}\bar{h}, \bar{x}\rangle}{\|\bar{x}+k\bar{h}\|^2} + \frac{k^2\langle \bar{Q}\bar{h}, \bar{h}\rangle}{\|\bar{x}+k\bar{h}\|^2}.$$

Taking limsup both sides of the latter inequality as $k \to +\infty$ and using the assumption (a_2), we obtain $\langle \bar{Q}\bar{h}, \bar{h}\rangle \geq 0$. Hence $\langle \bar{Q}\bar{h}, \bar{h}\rangle = 0$. By the assumption (a_1), we have $\bar{h} \in Null(\bar{Q})$. For any $x \in \mathcal{F}(\bar{p})$, by (2.14) and $\bar{h} \in Null(\bar{Q})$, one has

$$\begin{aligned}
&\langle \bar{Q}z^k + \bar{q}, x - z^k\rangle \\
=&\langle \bar{Q}(\bar{x}+k\bar{h}) + \bar{q}, x - \bar{x} - k\bar{h}\rangle \\
=&\langle \bar{Q}\bar{x} + \bar{q}, x - \bar{x}\rangle - k^2\langle \bar{Q}\bar{h}, \bar{h}\rangle + k\langle \bar{Q}\bar{h}, x - \bar{x}\rangle - k\langle \bar{Q}\bar{x} + \bar{q}, \bar{h}\rangle \\
\geq& - k\langle \bar{q}, \bar{h}\rangle \\
=& - k\langle \bar{Q}x + \bar{q}, \bar{h}\rangle \\
\geq& 0.
\end{aligned}$$

Hence $z^k \in S(\bar{p})$ for every k. This forces that $S(\bar{p})$ is unbounded, contrary to the assumption (b_2). Thus (b_3) holds.

$(b_3) \Rightarrow (b_1)$: Suppose that (b_3) holds. To obtain a contradiction, suppose that (b_1) does not hold, that is, there exists a sequence $\{p^k\} \subset \mathbb{P}$ such that $p^k \to \bar{p}$ and $S(p^k) = \emptyset$ for all k. By the assumption that $(QP(\bar{p}))$ satisfies (SCQ), we have $\mathcal{F}(\cdot)$ is lower semicontinuous at $\bar{p}$ (see Lemma 2.1). Then, for any $\bar{x} \in \mathcal{F}(\bar{p})$, for some $\epsilon > 0$, there exists k_0 such that $\mathcal{F}(p^k) \cap B(\bar{x}, \epsilon) \neq \emptyset$ for every $k \geq k_0$. Hence there exists i_0 such that

$$Z^{i,k} := \mathcal{F}(p^k) \cap \{x \in \mathbb{R}^n : \|x\| \leq i\} \tag{2.15}$$

is nonempty, compact and convex for every $k \geq k_0$ for every $i \geq i_0$. Applying Hartman-Stampacchia's theorem (see [51, Theorem 3.1]) for $VI(Q^k, q^k, Z^{i,k})$, we obtain that

$$SolVI(Q^k, q^k, Z^{i,k}) \neq \emptyset \ \ \forall i \geq i_0, \forall k \geq k_0.$$

Fix any $x^{i,k} \in SolVI(Q^k, q^k, Z^{i,k})$. Then

$$\langle Q^k x^{i,k} + q^k, z - x^{i,k}\rangle \geq 0 \ \ \forall z \in Z^{i,k}. \tag{2.16}$$

We now show that $\|x^{i,k}\| = i$. Indeed, suppose to the contrary that $\|x^{i,k}\| < i$. Then, there exists $\alpha > 0$ such that

$$\bar{B}(x^{k,i}, \alpha) := \{x \in \mathbb{R}^n : \|x - x^{i,k}\| \leq \alpha\} \subset \{x \in \mathbb{R}^n : \|x\| \leq i\}.$$

From (2.16),

$$\langle Q^k x^{i,k} + q^k, z - x^{i,k} \rangle \geq 0 \ \ \forall z \in \mathcal{F}(p^k) \cap \bar{B}(x^{k,i}, \alpha). \tag{2.17}$$

Since $\mathcal{F}(p^k)$ is convex and $\alpha > 0$, for each $z \in \mathcal{F}(p^k)$, there exists $t \in (0, 1)$ such that

$$z(t) := x^{i,k} + t(z - x^{i,k}) \in \mathcal{F}(p^k) \cap \bar{B}(x^{k,i}, \alpha).$$

Substituting $z(t)$ for z in (2.17), we obtain

$$0 \leq \langle Q^k x^{i,k} + q^k, z(t) - x^{i,k} \rangle = t\langle Q^k x^{i,k} + q^k, z - x^{i,k} \rangle.$$

This gives $\langle Q^k x^{i,k} + q^k, z - x^{i,k} \rangle \geq 0$ for all $z \in \mathcal{F}(p^k)$. Hence $x^{i,k} \in S(p^k)$, contrary to the assumption that $S(p^k) = \emptyset$ for all k. Thus $\|x^{i,k}\| = i$ for every $i \geq i_0$ and $k \geq k_0$.

Fix any $i \geq i_0$. Then, $\{x^{i,k}\}_{k \geq k_0}$, has a convergent subsequence. We suppose without loss of generality that $\lim_{k\to\infty} x^{i,k} = x^i$ for some $x^i \in \mathcal{F}(\bar{p})$ satisfying $\|x^i\| = i$. For each $z \in Z^i := \mathcal{F}(\bar{p}) \cap \{x \in \mathbb{R}^n : \|x\| \leq i\}$, since $\mathcal{F}(\cdot)$ is lower semicontinuous at $\bar{p}$, there exists $z^k \in Z^{i,k}$ such that $z^k \to z$. Passing (2.16) to limits as $k \to \infty$ yields

$$\langle \bar{Q} x^i + \bar{q}, z - x^i \rangle \geq 0. \tag{2.18}$$

Without loss of generality we may assume that $\|x^i\|^{-1} x^i \to \bar{h}$ for some $\bar{h} \in \mathbb{R}^n$ and $\|\bar{h}\| = 1$. Since $\{x^i\} \subset \mathcal{F}(\bar{p})$ and $\|x^i\| \to +\infty$ as $i \to +\infty$, $\bar{h} \in 0^+\mathcal{F}(\bar{p})$ (see Lemma 1.1).

It is easy to see that, for each $z \in \mathcal{F}(\bar{p})$, there exists $i_z \geq i_0$ such that $z \in Z^i$ for every $i \geq i_z$. From (2.18), $\langle \bar{Q} x^i + \bar{q}, z - x^i \rangle \geq 0$ for every $i \geq i_z$. This implies

$$\langle \bar{Q} x^i + \bar{q}, z \rangle \geq \langle \bar{Q} x^i, x^i \rangle + \langle \bar{q}, x^i \rangle. \tag{2.19}$$

Dividing both sides of (2.19) by $\|x^i\|^2$ and letting $i \to \infty$, one gets $\langle \bar{Q}\bar{h}, \bar{h}\rangle \le 0$. From the assumption (a_2) it follows $\langle \bar{Q}\bar{h}, \bar{h}\rangle \ge 0$. Hence $\langle \bar{Q}\bar{h}, \bar{h}\rangle = 0$. By the assumption (a_1), we have $h \in Null(\bar{Q})$.

Multiplying both sides of (2.19) by $\|x^i\|^{-1}$, letting $i \to \infty$ and using the assumption (a_2) gives $0 \ge \langle \bar{q}, \bar{h}\rangle$. From this and $h \in Null(\bar{Q})$ it follows that

$$\langle \bar{Q}z + \bar{q}, \bar{h}\rangle = \langle \bar{q}, \bar{h}\rangle \le 0.$$

Hence we obtain that $\bar{h} \in 0^+\mathcal{F}(\bar{p})$ and $\langle \bar{Q}z+\bar{q}, \bar{h}\rangle \le 0$ for every $z \in \mathcal{F}(\bar{p})$. This contradicts the assumption (b_3). The proof is complete. $\square$

Remark 2.5. *The assumption (a_2) is weaker than the assumption (ii) of [101, Theorem 2.3].*

The following example illustrates an application of Theorem 2.5. It also shows that Theorem 2.3 in [101] could not be applied for this problem.

Example 2.4. *Consider the problem $(QP(p))$ with $n = 2, m = 1$,*

$$\bar{Q} = \begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix}, \quad \bar{q} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \bar{Q}_1 = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}, \quad \bar{q}_1 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \bar{c}_1 = 0.$$

We have

$$\mathcal{F}(\bar{p}) = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \ge x_2^2\}$$

and

$$0^+\,\mathcal{F}(\bar{p}) = \{h = (h_1, h_2) \in \mathbb{R}^2 : h_1 \ge 0, h_2 = 0\}.$$

It is easy to check that (SCQ) holds. Since $\bar{Q}h = -2h_2 = 0$ for every $h = (h_1, h_2) \in 0^+\mathcal{F}(\bar{p})$, condition (a_1) is satisfied.

For each sequence $\{x^k = (x_1^k, x_2^k)\} \subset \mathcal{F}(\bar{p})$ satisfying $\|x^k\| \to \infty$, we consider the following two cases:

Case 1: $\{x_2^k\}$ is bounded. From $\|x^k\| \to \infty$ it follows $\|x_1^k\| \to \infty$. Then,

$$\limsup_{k\to\infty} \frac{(x^k)^T\bar{Q}x^k}{\|x^k\|^2} = \limsup_{k\to\infty} \frac{-2(x_2^k)^2}{(x_1^k)^2 + (x_2^k)^2} = 0.$$

Case 2: $\{x_2^k\}$ is unbounded. Then,

$$\limsup_{k\to\infty} \frac{(x^k)^T \bar{Q} x^k}{\|x^k\|^2} \geq \limsup_{k\to\infty} \frac{-2(x_2^k)^2}{(x_2^k)^4 + (x_2^k)^2} = 0.$$

We see that, in both cases, condition (a_2) is satisfied.

Let $\bar{x} = (\bar{x}_1, \bar{x}_2) = (1/2, 1/2) \in \mathcal{F}(\bar{p})$. Then,

$$(\bar{Q}\bar{x} + \bar{q})^T h = h_1 + h_2 > 0 \ \ \forall h = (h_1, h_2) \in 0^+\mathcal{F}(\bar{p}) \setminus \{0\}.$$

This leads to that (b_3) holds.

By Theorem 2.5, we deduce that (b_1), (b_2), and (b_4) hold.

On the other hand, let $x^k = (k^3, k) \in \mathcal{F}(\bar{p})$ for $k = 1, 2, \ldots$ We have

$$(x^k)^T \bar{Q} x^k = -2k^2 \to -\infty.$$

This implies that condition (ii) in [101, Theorem 2.3] is not satisfied. Hence [101, Theorem 2.3] could not be applied for the problem in this example.

2.4. Conclusions

We have used the obtained results on solution existence to investigate continuity of the global optimal solution map (Theorems 2.1 and 2.2) and lower semicontinuity of the local optimal solution map of parametric QCQP problems (Theorem 2.3). The upper semicontinuity of the stationary solution map has been also investigated (Theorem 2.4). Using the tool related to parametric variational inequality, we have proposed the assertions which are equivalent to the emptiness and boundedness of the stationary solution set (Theorem 2.5).

Chapter 3

Continuity and directional differentiability of the optimal value function

This chapter deals with continuity and directional differentiability of the optimal value function in nonconvex QCQP problems. Among our proposed assumptions, there are some weaker than the assumptions used in the cited works (applied for QP). In Section 3.1, continuity of optimal value function is studied. Sections 3.2 and 3.3 establish sufficient conditions for the first- and the second-order directional differentiability of the optimal value function.

This chapter is written on the basis of the results in [75, 102].

3.1. Continuity of the optimal value function

The following theorem shows the necessary and sufficient condition for continuity of the optimal value function.

Theorem 3.1. *Consider the problem* $(QP(p))$ *and* $\bar{p} \in \mathbb{P}$. *Assume that* f *is bounded from below over* $\mathcal{F}(\bar{p}) \neq \emptyset$. *Then,* φ *is continuous at* $\bar{p}$ *if and only if* (SCQ) *and* (A_3) *are fulfilled at* $\bar{p}$.

Proof. Necessity: Firstly, we show that (SCQ) is satisfied at $\bar{p}$. Indeed, suppose that φ is continuous at $\bar{p}$ and (SCQ) is not satisfied at $\bar{p}$. By Lemma 2.1, there exists $p^k \to \bar{p}$ such that, for each k, the system $g_i(x, p^k) \leq 0,\ i = 1, \ldots, m$, has no solution, that is, $\mathcal{F}(p^k) = \emptyset$. Then, $\varphi(p^k) = +\infty$. Since φ is continuous at $\bar{p}$, we get $\varphi(\bar{p}) = +\infty$, that is $\mathcal{F}(\bar{p}) = \emptyset$. We have arrived at a contradiction. This shows that (SCQ) is fulfilled at $\bar{p}$.

It remains to show that (A_3) holds at $\bar{p}$. Indeed, suppose that φ is continuous at $\bar{p}$ but (A_3) fails to hold at $\bar{p}$. Then, there exists $\bar{v} \in (0^+\mathcal{F}(\bar{p})) \setminus \{0\}$ such that $\bar{v}^T Q \bar{v} \leq 0$. Since $\mathcal{F}(\bar{p}) \neq \emptyset$, there exists $x^0 \in \mathcal{F}(\bar{p})$ such that $x^0 + t\bar{v} \in \mathcal{F}(\bar{p})$ for every $t > 0$. Let $Q^k = \bar{Q} - \frac{1}{k} I$ and let $p^k = (Q^k, \bar{q}, \bar{Q}_1, \bar{q}_1, \bar{c}_1, \ldots, \bar{Q}_m, \bar{q}_m, \bar{c}_m) \to \bar{p}$. Then, we obtain that $\bar{v}^T Q^k \bar{v} = \bar{v}^T(\bar{Q} - \frac{1}{k} I)\bar{v} < 0$. Hence

$$\begin{aligned} f(x^0 + t\bar{v}, p^k) =& \frac{1}{2}(x^0 + t\bar{v})^T Q^k (x^0 + t\bar{v}) + \bar{q}^T (x^0 + t\bar{v}) \\ =& \frac{1}{2} t^2 \bar{v}^T Q^k \bar{v} + t(Q^k x^0 + \bar{q})\bar{v} + f(x^0, \bar{p}) \to -\infty \end{aligned}$$

as $t \to +\infty$. Therefore $\varphi(p^k) = -\infty$. Since φ is continuous at $\bar{p}$, we have $\varphi(\bar{p}) = -\infty$. This contradicts the assumption that f is bounded from below over $\mathcal{F}(\bar{p}) \neq \emptyset$. Thus (A_3) is fulfilled at $\bar{p}$.

Sufficiency: Suppose that (SCQ) and (A_3) hold at $\bar{p}$. We have to prove that $\varphi(p^k) \to \varphi(\bar{p})$ for every sequence $\{p^k\} \subset \mathbb{P}$ satisfying $p^k \to \bar{p}$, By Lemma 2.1, there exists $k_0 > 0$ such that $\mathcal{F}(p^k) \neq \emptyset$ for every $k \geq k_0$. From the assumption (A_3) it follows $\bar{p} \in G$. Combining this and Lemma 2.2, we have S is open. Hence there exists $k_1 > 0$ such that $p^k \in G$ for every $k \geq k_1$. By Corollary 1.3, we obtain $G(p^k) \neq \emptyset$, that is, there exists $x^k \in \mathcal{F}(p^k)$ such that $\varphi(p^k) = f(x^k, p^k)$.

From Corollary 1.3 it follows that $G(\bar{p}) \neq \emptyset$. Hence there exists $x^0 \in G(\bar{p})$ such that $\varphi(\bar{p}) = f(x^0, \bar{p})$. From Lemma 2.1, the set-valued map $p \mapsto \mathcal{F}(p)$ is lower semicontinuous at $\bar{p}$. There thus exists $y^k \in \mathcal{F}(p^k)$

such that $y^k \to x^0$. We have

$$\limsup_{k\to\infty} \varphi(p^k) \le \limsup_{k\to\infty} f(y^k, \bar{p}) = f(x^0, \bar{p}) = \varphi(\bar{p}.) \tag{3.1}$$

We now claim that the sequence $\{x^k\}_{k\ge k_1}$ is bounded. Indeed, if $\{x^k\}_{k\ge k_1}$ is unbounded then, by taking a subsequence if necessary, we may assume that $\|x^k\| \to +\infty$ as $k \to \infty$ and $\|x^k\| \ne 0$ for every $k \ge 1$. Without loss of generality, we may assume that $x^k/\|x^k\| \to h$. Since $x^k \in \mathcal{F}(p^k)$, $g_i(x^k, p^k) \le 0, i = 1, \ldots, m$. We obtain $h \in (0^+\mathcal{F}(\bar{p})) \setminus \{0\}$ (see Lemma 1.1).

On the other hand, since $x^k \in G(p^k)$ and $y^k \in \mathcal{F}(p^k)$, one gets

$$\frac{1}{2}(y^k)^T Q^k y^k + (q^k)^T y^k \ge \frac{1}{2}(x^k)^T Q^k x^k + (q^k)^T x^k.$$

Dividing both of sides of the above inequality by $\|x^k\|^2$ and taking $k \to \infty$ yields $h^T \bar{Q} h \le 0$. This contradicts the assumption (A_3). Hence $\{x^k\}$ is bounded.

Without loss of generality, we may assume that $x^k \to \bar{x}$. Since $x^k \in \mathcal{F}(p^k)$, $g_i(x^k, p^k) \le 0, i = 1, \ldots, m$. Letting $k \to \infty$, we obtain that $g_i(\bar{x}, \bar{p}) \le 0, i = 1, \ldots, m$, that is, $\bar{x} \in \mathcal{F}(\bar{p})$. Hence

$$\lim_{k\to\infty} \varphi(p^k) = \lim_{k\to\infty} f(x^k, p^k) = f(\bar{x}, \bar{p}) \ge \varphi(\bar{p}). \tag{3.2}$$

Combining (3.1) with (3.2) yields $\lim_{k\to\infty} \varphi(p^k) = \varphi(\bar{p})$. This finishes the proof. $\square$

Remark 3.1. *The assumption (A_3) of Theorem 3.1 is weaker than the uniform compactness of $\mathcal{F}(p)$ near $\bar{p}$ used in [36, Theorem 3.3].*

The following result characterizes the stability and the Lipschitzian stability for parametric nonconvex QCQP problem.

Theorem 3.2. *Consider the problem $(QP(p))$. Assume that (SCQ) and (A_3) hold at $\bar{p} = (\bar{Q}, \bar{q}, \bar{Q}_1, \bar{q}_1, \bar{c}_1, \ldots, \bar{Q}_m, \bar{q}_m, \bar{c}_m) \in \mathbb{P}$. Then, the following four statements are equivalent:*

(a) $G(\cdot)$ is lower semicontinuous at $\bar{p}$;

(b) $G(\cdot)$ is continuous at $\bar{p}$;

(c) $G(\bar{p})$ is a singleton and $\varphi(\cdot)$ is locally Lipschitz at $\bar{p}$;

(d) $G(\bar{p})$ is a singleton and $\varphi(\cdot)$ is continuous at $\bar{p}$.

Proof. Suppose that (a) holds. According to Lemma 2.1, $G(\cdot)$ is upper semicontinuous at $\bar{p}$ under assumptions (SCQ) and (A_3). Hence the global optimal solution map $G(\cdot)$ is continuous at $\bar{p}$, that is (b) is satisfied.

Next, we prove that (b) *implies* (c). Indeed, suppose that the global optimal solution map $G(\cdot)$ is continuous at $\bar{p}$. From Lemma 4.3 it follows that $G(\bar{p})$ is a singleton. It remains to show that $\varphi(\cdot)$ is Lipschitz continuous around $\bar{p}$. Since $f(.,.)$ is continuously differentiable, there exists $\delta > 0$ such that $f(.,.)$ is Lipschitz continuous with modulus L_f on the neighborhood $U_\delta(\bar{x}, \bar{p})$.

By the assumption (SCQ) and Remark 2.2, the feasible set map $p \mapsto \mathcal{F}(p)$ is locally Lipschitz-like around $(\bar{p}, \bar{x})$ for some $\bar{x} \in \mathcal{F}(\bar{p})$ (see, for instance, [30, Example 4D.3]), that is, there exist three positive real numbers ϵ, γ, and $L_{\mathcal{F}}$ such that

$$\mathcal{F}(p^1) \cap U_\gamma(\bar{x}) \subset \mathcal{F}(p^2) + L_{\mathcal{F}} \|p^2 - p^1\| B_{\mathbb{P}}(0,1) \tag{3.3}$$

for every $p^1, p^2 \in U_\epsilon(\bar{p})$.

Without loss of generality, we may assume that $(2L_{\mathcal{F}}+1)\epsilon+\gamma < \delta$. Let any $p^1, p^2 \in U_\epsilon(\bar{p})$. By the assumption (SCQ) and Lemma 2.1, the set-valued map $p \mapsto \mathcal{F}(p)$ is lower semicontinuous at $\bar{p}$. This implies $\mathcal{F}(p^1) \neq \emptyset$, for $\epsilon > 0$ small enough. On the other hand, from (A_3) it follows $\bar{p} \in G$. Since S is open, there exists $\epsilon > 0$ small enough such that $p^1 \in G$. By Corollary 1.3, we have $G(p^1) \neq \emptyset$, for $\epsilon > 0$ small enough. Thus there exists $x^1 \in G(p^1)$.

Since $G(\bar{p})$ is a singleton, $G(\cdot)$ is lower semicontinuous at $\bar{p}$. Hence we can assume that $\epsilon > 0$ small enough to guarantee that $x^1 \in U_\gamma(\bar{x})$. It leads to $x^1 \in G(p^1) \cap U_\gamma(\bar{x}) \subset \mathcal{F}(p^1) \cap U_\gamma(\bar{x})$. Due to (3.3), there exists $x^2 \in \mathcal{F}(p^2)$ such that

$$\|x^2 - x^1\| \leq L_{\mathcal{F}} \|p^2 - p^1\|.$$

We obtain that

$$\|(x^1, p^1) - (\bar{x}, \bar{p})\| \leq \|x^1 - \bar{x}\| + \|p^1 - \bar{p}\| < \gamma + \epsilon < \delta$$

and

$$\begin{aligned}
\|(x^2, p^2) - (\bar{x}, \bar{p})\| &\leq \|x^2 - \bar{x}\| + \|p^2 - \bar{p}\| \\
&\leq \|x^2 - x^1\| + \|x^1 - \bar{x}\| + \|p^2 - \bar{p}\| \\
&< L_{\mathcal{F}} \|p^2 - p^1\| + \gamma + \epsilon \\
&< 2L_{\mathcal{F}}\epsilon + \gamma + \epsilon \\
&= (2L_{\mathcal{F}} + 1)\epsilon + \gamma \\
&< \delta.
\end{aligned}$$

Then, $(x^1, p^1), (x^2, p^2) \in U_\delta(\bar{x}, \bar{p})$.

Since $\varphi(p^2) \leq f(x^2, p^2)$ and $\varphi(p^1) = f(x^1, p^1)$, it implies

$$\begin{aligned}
\varphi(p^2) - \varphi(p^1) &\leq f(x^2, p^2) - f(x^1, p^1) \\
&\leq |f(x^2, p^2) - f(x^1, p^1)| \\
&\leq L_f \|(x^2, p^2) - (x^1, p^1)\| \\
&\leq L_f (\|x^2 - x^1\| + \|p^2 - p^1\|) \\
&\leq L_f (L_{\mathcal{F}} \|p^2 - p^1\| + \|p^2 - p^1\|) \\
&= L_f (L_{\mathcal{F}} + 1) \|p^2 - p^1\|.
\end{aligned} \tag{3.4}$$

Changing the role of p^1 and p^2, one has

$$\varphi(p^1) - \varphi(p^2) \leq L_f (L_{\mathcal{F}} + 1) \|p^1 - p^2\|. \tag{3.5}$$

From (3.4) and (3.5) it follows that $\varphi(\cdot)$ is Lipschitz continuous around $\bar{p}$ with modulus $L_f(L_{\mathcal{F}} + 1)$.

Clearly, (c) *implies* (d).

By Theorem 2.2, we conclude that (d) *implies* (a). The proof is then complete. $\square$

Example 3.1. *Consider again the problem in Example 2.1. Both* (SCQ) *and* (A_3) *hold at* $\bar{p}$. *It is easy to check that* $G(\bar{p}) = \{\bar{x} = (-1, 0)\}$. *By Theorem 3.2, the global optimal solution map* $G(\cdot)$ *is continuous at* $\bar{p}$ *and* φ *is Lipschitz continuous around* $\bar{p}$.

3.2. First-order directional differentiability

The main results in this section will describe sufficient conditions for the first-order directional differentiability and give explicit formulas for computing this directional derivative of the optimal value function φ of the problem $(QP(p))$.

For each $\lambda = (\lambda_1, ..., \lambda_m) \in \mathbb{R}^n_+$, *the Lagrange function* $L(x, p, \lambda)$ is defined by

$$L(x, p, \lambda) = f(x, p) + \sum_{i=1}^{m} \lambda_i g_i(x, p).$$

For $p^0 = (Q^0, q^0, Q_1^0, q_1^0, c_1^0, \ldots, Q_m^0, q_m^0, c_m^0) \in \mathbb{P}$ and $h \in \mathbb{R}^n$, we have

$$\nabla L_{(x,p)}(\bar{x}, p, \lambda)^T (h, p^0) = (Q\bar{x}+q)^T h + \sum_{i=1}^{m} \lambda_i (Q_i \bar{x} + q_i)^T h + \\ + f(x, p^0) + \sum_{i=1}^{m} \lambda_i g_i(\bar{x}, p^0)$$

and

$$(h, p^0)^T \nabla^2 L_{(x,p)}(\bar{x}, p, \lambda)(h, p^0) = h^T \Big(Q + \sum_{i \in I(\bar{x},p)} \lambda_i Q_i \Big) h + \\ + 2\Big(Q^0 \bar{x} + q^0 + \sum_{i \in I(\bar{x},p)} \lambda_i (Q_i^0 \bar{x} + q_i^0) \Big)^T h.$$

For $\bar{x} \in G(p)$, denote by $\Lambda(\bar{x}, p)$ the set of all Lagrange multipliers corresponding to $\bar{x}$.

We consider the following assumption

Assumption $(\mathbf{A}_4)$ *For every* $t_k \downarrow 0$, *for every* $x^k \in G(p + t_k p^0)$ *satisfying*

$x^k \to \bar{x} \in G(p)$, *and for every* $\lambda \in \Lambda(\bar{x}, p)$, *the following inequality holds*

$$\liminf_{k\to+\infty} \frac{(x^k - \bar{x})^T\Big(Q + \sum_{i\in I(\bar{x},p)} \lambda_i Q_i\Big)(x^k - \bar{x})}{t_k} \geq 0. \tag{3.6}$$

Remark 3.2. *If* $\nabla^2_{xx} L(\bar{x}, p, \lambda) = Q + \sum_{i\in I(\bar{x},p)} \lambda_i Q_i$ *is a positive semidefinite matrix, then* (A_4) *holds. Particularly, from the assumption that* Q_i, $i = 1, \ldots, m$, *are positive semidefinite matrices it follows that* (A_4) *holds if* Q *is positive semidefinite.*

Remark 3.3. (A_4) *holds if the sequence* $\{\|x^k - \bar{x}\|/t_k\}$ *is bounded.*

Now, we describe a general situation where (A_4) is fulfilled.

Proposition 3.1. *Assume that* $(QP(p))$ *satisfies (SCQ) and*

$$v^T\Big(Q + \sum_{i\in I(\bar{x},p)} \lambda_i Q_i\Big)v > 0 \ \ \forall v \in C(\bar{x}) \setminus \{0\}, \ \forall \lambda \in \Lambda(\bar{x}, p), \tag{3.7}$$

where $C(\bar{x}) := \{v \in \mathbb{R}^n : (Q\bar{x} + q)^T v \leq 0, (Q_i\bar{x} + q_i)^T v \leq 0, i \in I(\bar{x}, p)\}$. *Then,* (A_4) *holds.*

Proof. On the contrary, suppose that (A_4) does not hold, that is, there exist $\lambda \in \Lambda(\bar{x}, p)$, a convergent to zero sequence $\{t_k\}$ and $x^k \in G(p+t_kp^0)$ tending to $\bar{x} \in G(p)$ satisfying

$$\lim_{k\to+\infty} \frac{(x^k - \bar{x})^T\Big(Q + \sum_{i\in I(\bar{x},p)} \lambda_i Q_i\Big)(x^k - \bar{x})}{t_k} < 0. \tag{3.8}$$

By taking a subsequence, if necessary, we may assume that

$$(x^k - \bar{x})^T\Big(Q + \sum_{i\in I(\bar{x},p)} \lambda_i Q_i\Big)(x^k - \bar{x}) < 0, \tag{3.9}$$

$$\lim_{k\to+\infty} \frac{\|x^k - \bar{x}\|}{t_k} = +\infty, \ \|x^k - \bar{x}\| \neq 0 \ \forall k. \tag{3.10}$$

Without loss of generality, assume that $(x^k - \bar{x})/\|x^k - \bar{x}\| \to \bar{v} \neq 0$. Dividing both sides of (3.9) by $\|x^k - \bar{x}\|^2$ and letting $k \to \infty$ yields

$$\bar{v}^T \left(Q + \sum_{i \in I(\bar{x},p)} \lambda_i Q_i \right) \bar{v} \leq 0. \tag{3.11}$$

For every $i \in I(\bar{x}, p)$, we have

$$\begin{aligned}
0 \geq\, & g_i(x^k, p + t_k p^0) - g_i(\bar{x}, p) \\
=& g_i(x^k, p) - g_i(\bar{x}, p) + t_k g_i(x^k, p^0) \\
=& \frac{1}{2}(x^k - \bar{x})^T Q_i (x^k - \bar{x}) + (Q_i \bar{x} + q_i)^T (x^k - \bar{x}) + t_k g_i(x^k, p^0).
\end{aligned}$$

Due to $Q_i,\ i \in I(\bar{x}, p)$, are positive semidefinite, we get

$$(Q_i \bar{x} + q_i)^T (x^k - \bar{x}) + t_k g_i(x^k, p^0) \leq 0. \tag{3.12}$$

Dividing both sides of the last inequality by $\|x^k - \bar{x}\|$ and letting $k \to \infty$ yields

$$(Q_i \bar{x} + q_i)^T \bar{v} \leq 0. \tag{3.13}$$

We now show that $(Q\bar{x} + q)^T \bar{v} \leq 0$. Indeed, we obtain that

$$\begin{aligned}
& \varphi(p + t_k p^0) - \varphi(p) \\
=\ & f(x^k, p + t_k p^0) - f(\bar{x}, p) \\
=\ & f(x^k, p) - f(\bar{x}, p) + t_k f(x^k, p^0) \\
=\ & \tfrac{1}{2}(x^k - \bar{x})^T Q (x^k - \bar{x}) + (Q\bar{x} + q)^T (x^k - \bar{x}) + t_k f(x^k, p^0).
\end{aligned} \tag{3.14}$$

By Lemma 2.1, the set-valued map $\mathcal{F}(\cdot)$ is lower semicontinuous at p. Hence there exists $\bar{u} \in \mathbb{R}^n$ such that, for t_k small enough, we have $\bar{x} + t_k \bar{u} \in \mathcal{F}(p + t_k p^0)$. This implies that

$$\begin{aligned}
& \varphi(p + t_k p^0) - \varphi(p) \\
\leq& f(\bar{x} + t_k \bar{u}, p + t_k p^0) - f(\bar{x}, p) \\
=& f(\bar{x} + t_k \bar{u}, p) - f(\bar{x}, p) + t_k f(\bar{x} + t_k \bar{u}, p^0) \\
=& \frac{1}{2}(t_k)^2 \bar{u}^T Q \bar{u} + t_k (Q\bar{x} + q)^T \bar{u} + t_k f(\bar{x} + t_k \bar{u}, p^0).
\end{aligned}$$

Hence

$$\begin{array}{rl} & \frac{1}{2}(x^k-\bar{x})^TQ(x^k-\bar{x})+(Q\bar{x}+q)^T(x^k-\bar{x})+t_kf(x^k,p^0) \\ \leq & \frac{1}{2}(t_k)^2\bar{u}^TQ\bar{u}+t_k(Q\bar{x}+q)^T\bar{u}+t_kf(\bar{x}+t_k\bar{u},p^0). \end{array} \tag{3.15}$$

Dividing both sides of the last inequality by $\|x^k-\bar{x}\|$ and letting $k \to \infty$ yields $(Q\bar{x}+q)^T\bar{v} \leq 0$. Combining this with (3.13) yields $\bar{v} \in C(\bar{x})$. By (3.11), we have a contradiction. The desired conclusion follows. □

We say that *the Mangasarian-Fromovitz under direction* $p^0 \in \mathbb{P}$ *Regularity Condition* $(MFRC)_{p^0}$ holds at $\bar{x}$ if there exists $v^0 \in \mathbb{R}^n$ such that

$$(Q_i\bar{x}+q_i)^Tv^0+g_i(\bar{x},p^0)<0 \ \ \forall i \in I(\bar{x},p).$$

It follows immediately that $(MFRC)_{p^0}$ is weaker than $(MFCQ)$ (see, for instance, [5, 39]).

To characterize directional differentiability of the optimal value function, Auslender and Cominetti [5] proposed Condition $(SOSC)_{p^0}$. Applying $(SOSC)_{p^0}$ for each global optimal solution $\bar{x}$ of $(QP(p))$, we have the following condition:

$$(SOSC)_{p^0} \ \left\{ \begin{array}{l} \textit{For each critical direction } v \in C(\bar{x}), \textit{ one has} \\ \sup\Big\{v^T\Big(Q+\sum_{i\in I(\bar{x},p)}\lambda_iQ_i\Big)v : \lambda \in \Lambda^*(\bar{x},p)\Big\} > 0, \end{array}\right.$$

where $\Lambda^*(\bar{x},p^0)$ is the global optimal solution set of the following problem

$$\sup_{\lambda\in\Lambda(\bar{x},p)}\Big\{f(\bar{x},p^0)+\sum_{i=1}^m\lambda_ig_i(\bar{x},p^0)\Big\}.$$

Now we shall show that (A_4) is weaker than $(SOSC)_{p^0}$ applied for $(QP(p))$.

Proposition 3.2. *Under the assumption* $(MFRC)_{p^0}$, *assume that for an arbitrary sequence* $\{t_k\}$ *converging to zero and for a* $x^k \in G(p+t_kp^0)$, *the sequence* $\{x^k\}$ *converges to* $\bar{x} \in G(p)$. *If* $(SOSC)_{p^0}$ *holds at* $\bar{x}$, *then the sequence* $\{\|x^k-\bar{x}\|/t_k\}$ *is bounded. Furthermore,* (A_4) *holds.*

Proof. By the assumption and by [5, Proposition 2], we deduce that the sequence $\{(x^k - \bar{x})/t_k\}$ is bounded. From Remark 3.3 it follows that (A_4) holds. $\square$

The Linear Independence Constraint Qualification (LICQ) is satisfied at $\bar{x} \in \mathcal{F}(p)$ if the vectors $Q_i\bar{x} + q_i, i \in I(\bar{x}, p)$, are linearly independent. It is well-known that $(MFCQ)$ is weaker than $(LICQ)$ (see, for instance, [64, p.103–104]).

Condition $(H3)$ in [71] applied for the global optimal solution $\bar{x}$ of $(QP(p))$ is stated as follows:

$(H3)$: *For every sequence* $\{t_k\}$ *satisfying* $t_k \downarrow 0$, *and for every sequence* $\{x^k\}$ *with* $x^k \in G(p + t_k p^0)$ *such that* $x^k \to \bar{x} \in G(p)$, *the following inequality is satisfied*

$$\limsup_{k\to+\infty} \frac{\|x^k - \bar{x}\|^2}{t_k} < +\infty.$$

The following proposition shows that, in some cases, $(H3)$ is stronger than (A_4).

Proposition 3.3. *Consider* $(QP(p))$. *Assume that one of the following conditions is satisfied:*

i) (LICQ) holds;

ii) $Q_i = 0$, *for every* $i = 1, \dots, m$;

iii) f *is convex and* (SCQ) *holds.*

Then, $(H3)$ *implies* (A_4).

Proof. Suppose that $(H3)$ holds, i.e., for every sequence $\{t_k\}, t_k \downarrow 0$, and for every sequence $\{x^k\}, x^k \in G(p + t_k p^0)$ such that $x^k \to \bar{x} \in G(p)$, the following inequality is satisfied

$$\limsup_{k\to+\infty} \frac{\|x^k - \bar{x}\|^2}{t_k} < +\infty. \tag{3.16}$$

By taking a subsequence if necessary, we may assume that

$$\liminf_{k\to+\infty} \frac{(x^k-\bar{x})^T\left(Q+\sum_{i\in I(\bar{x},p)}\lambda_i Q_i\right)(x^k-\bar{x})}{t_k}$$
$$=\lim_{k\to+\infty} \frac{(x^k-\bar{x})^T\left(Q+\sum_{i\in I(\bar{x},p)}\lambda_i Q_i\right)(x^k-\bar{x})}{t_k}.$$

From (3.16) it follows that $\{\|x^k-\bar{x}\|/(t_k)^{\frac{1}{2}}\}$ is bounded. Without loss of generality, we may assume that $(x^k-\bar{x})/(t_k)^{\frac{1}{2}} \to \bar{w}$ for some $\bar{w}\in\mathbb{R}^n$. Since $x^k \in G(p+t_kp^0)$, $g_i(x^k, p+t_kp^0) \le 0$ for every $i\in I(\bar{x},p)$. By (3.12), we get

$$(Q_i\bar{x}+q_i)^T(x^k-\bar{x})+t_kg_i(\bar{x},p^0)\le 0.$$

Dividing both sides of the last inequality by $(t_k)^{\frac{1}{2}}$ and letting $k\to\infty$ yields

$$(Q_i\bar{x}+q_i)^T\bar{w}\le 0. \tag{3.17}$$

By (3.15), we obtain that

$$\begin{aligned}&\frac{1}{2}(x^k-\bar{x})^TQ(x^k-\bar{x})+(Q\bar{x}+q)^T(x^k-\bar{x})+t_kf(\bar{x},p^0)\\ \le&\frac{1}{2}(t_k)^2\bar{u}^TQ\bar{u}+t_k(Q\bar{x}+q)^T\bar{u}+t_kf(\bar{x},p^0)\end{aligned}$$

for some $\bar{u}\in\mathbb{R}^n$. Dividing both sides of the above inequality by $(t_k)^{\frac{1}{2}}$ and letting $k\to\infty$ yields

$$(Q\bar{x}+q)^T\bar{w}\le 0. \tag{3.18}$$

From (3.17) and (3.18) it follows $\bar{w}\in C(\bar{x})$. By the necessary second order optimality condition (see [7, Theorem 1.2] and the references therein) and assumptions (i)–(iii), we have

$$\bar{w}^T\left(Q+\sum_{i\in I(\bar{x},p)}\lambda_iQ_i\right)\bar{w}\ge 0.$$

Therefore

$$\liminf_{k\to+\infty} \frac{(x^k-\bar{x})^T\left(Q+\sum_{i\in I(\bar{x},p)}\lambda_i Q_i\right)(x^k-\bar{x})}{t_k}$$
$$=\bar{w}^T\left(Q+\sum_{i\in I(\bar{x},p)}\lambda_i Q_i\right)\bar{w}\geq 0.$$

The proposition is proved. $\square$

The set-valued map $G:\tilde{p}\mapsto G(\tilde{p})$ is said to be *upper pseudo Lipschitzian* (or *calm*) at a point $(p,\bar{x})$, where $\bar{x}\in G(p)$, if there exist neighborhoods $V(p)$ and $V(\bar{x})$ of the points $p,\bar{x}$ and a number $l>0$ such that

$$G(\tilde{p})\cap V(\bar{x})\subset G(p)+l\|\tilde{p}-p\|B(0,1)$$

for all $\tilde{p}\in V(p)$ (see [72]).

Assumption (A_4) is weaker than the assumption of the calmness of the global optimal solution map $G:\tilde{p}\mapsto G(\tilde{p})$ applied for $QP(p)$. This is given below.

Proposition 3.4. *Consider the problem $(QP(p))$ and assume that for an arbitrary sequence $\{t_k\}$ converging to zero and for a $x^k\in G(p+t_kp^0)$, the sequence $\{x^k\}$ converges to $\bar{x}\in G(p)$. If $G(\cdot)$ is calm at a point $(p,\bar{x})$, then the sequence $\{\|x^k-\bar{x}\|/t_k\}$ is bounded. Furthermore, (A_4) holds.*

Proof. Since $G(\cdot)$ is calm at a point $(p,\bar{x})$, there exists a number $k_0>0$ such that $\|x^k-\bar{x}\|\leq lt_k\|p^0\|$ for every $k\geq k_0$. Hence

$$\limsup_{k\to+\infty}\frac{\|x^k-\bar{x}\|}{t_k}<+\infty,$$

that is, $\{(x^k-\bar{x})/t_k\}$ is bounded. From Remark 3.3 it follows that (A_4) holds. $\square$

To prove the main results we need the following lemmas.

Lemma 3.1. *Assume that $(QP(p))$ satisfies (SCQ) and (A_3). Then, for each sequence $\{p^k\} \subset \mathbb{P}$ converging to p and $x^k \in G(p^k)$, there exists a subsequence $\{x^{k^i}\}$ of $\{x^k\}$ such that $x^{k^i} \to \bar{x} \in G(p)$.*

Proof. Assume that $\{p^k\} \subset \mathbb{P}$ converges to p and $\{x^k\} \subset G(p^k)$. Let any $x \in \mathcal{F}(p)$. By Lemma 2.1, the set-valued map $\tilde{p} \mapsto \mathcal{F}(\tilde{p})$ is lower semicontinuous at p. Thus there exists $\{y^k\} \subset \mathcal{F}(p^k)$ such that $y^k \to x$.

Suppose that $\{x^k\}$ is unbounded. Without loss of generality, we may assume that $\|x^k\| \to \infty$ and $x^k/\|x^k\| \to \bar{v}$ as $k \to \infty$ for some $\bar{v} \in \mathbb{R}^n$. Since $x^k \in \mathcal{F}(p^k)$, $g_i(x^k, p^k) \leq 0$, for every $i = 1, \ldots, m$. By Lemma 1.1, we get $\bar{v} \in (0^+\mathcal{F}(p)) \setminus \{0\}$.

Since $x^k \in G(p^k)$, we obtain that

$$\frac{1}{2}(x^k)^T Q^k x^k + (q^k)^T x^k \leq \frac{1}{2}(y^k)^T Q^k y^k + (q^k)^T y^k. \tag{3.19}$$

Dividing both sides of the inequality (3.19) by $\|x^k\|^2$ and letting $k \to \infty$, we get $\bar{v}^T Q \bar{v} \leq 0$. This contradicts the assumption (A_3). Hence there exists a subsequence $\{x^{k_j}\} \subset \{x^k\}$ such that $x^{k_j} \to \bar{x}$.

From (3.19) it follows that

$$\frac{1}{2}(x^{k_j})^T Q^{k_j} x^{k_j} + (q^{k_j})^T x^{k_j} \leq \frac{1}{2}(y^{k_j})^T Q^{k_j} y^{k_j} + (q^{k_j})^T y^{k_j}$$

and

$$g_i(x^{k_j}, p^{k_j}) \leq 0, \ \forall i = 1, \ldots, m.$$

Take limits in the above inequalities as $j \to \infty$, we obtain that

$$\frac{1}{2}\bar{x}^T Q \bar{x} + q^T \bar{x} \leq \frac{1}{2} x^T Q x + q^T x$$

and

$$g_i(\bar{x}, p) \leq 0, \ \forall i = 1, \ldots, m.$$

These lead to $\bar{x} \in G(p)$. $\square$

Denote

$$D(\bar{x}, p, p^0) := \{v \in \mathbb{R}^n : (Q_i \bar{x} + q_i)^T v + g_i(\bar{x}, p^0) \leq 0, i \in I(\bar{x}, p)\}.$$

Next, we show that $D(\bar{x}, p, p^0)$ is nonempty.

Lemma 3.2. *If* $(QP(p))$ *satisfies (SCQ) then* $D(\bar{x}, p, p^0)$ *is nonempty.*

Proof. Due to Remark 2.2, $(MFCQ)$ holds at $\bar{x}$. It follows that there exists $v^0 \in \mathbb{R}^n$ such that $(Q_i\bar{x} + q_i)^T v^0 < 0 \quad \forall i \in I(\bar{x}, p)$. For each $i \in I(\bar{x}, p)$, there exists $m_i > 0$ large enough such that

$$(Q_i\bar{x} + q_i)^T (m_i v^0) + g_i(\bar{x}, p^0) \leq 0.$$

Let $m = max\{m_i, i \in I(\bar{x}, p)\}$. We have

$$(Q_i\bar{x} + q_i)^T (m v^0) + g_i(\bar{x}, p^0) \leq 0 \;\; \forall i \in I(\bar{x}, p),$$

i.e., $D(\bar{x}, p, p^0)$ is nonempty. □

Consider the following two linear programs

$$\inf_{v \in D(\bar{x}, p, p^0)} \left\{ (Q\bar{x} + q)^T v + f(\bar{x}, p^0) \right\} \tag{3.20}$$

and

$$\sup_{\lambda \in \Lambda(\bar{x}, p)} \left\{ f(\bar{x}, p^0) + \sum_{i=1}^{m} \lambda_i g_i(\bar{x}, p^0) \right\}. \tag{3.21}$$

The global optimal solution sets of problems (3.20) and (3.21) will be denoted by $D^*(\bar{x}, p, p^0)$ and $\Lambda^*(\bar{x}, p)$, respectively.

Applying [5, Lemma 2] for the problem $(QP(p))$ gives the following lemma.

Lemma 3.3. *If either* $\Lambda(\bar{x}, p)$ *or* $D(\bar{x}, p, p^0)$ *are nonempty, then*

$$\inf_{v \in D(\bar{x}, p, p^0)} \left\{ (Q\bar{x} + q)^T v + f(\bar{x}, p^0) \right\} = \sup_{\lambda \in \Lambda(\bar{x}, p)} \left\{ f(\bar{x}, p^0) + \sum_{i=1}^{m} \lambda_i g_i(\bar{x}, p^0) \right\}.$$

Moreover, if both feasible sets are nonempty, then the extrema are attained.

By the results in [35] and Lemmas 3.2 and 3.3, the extrema of the problems (3.20) and (3.21) are attained and the common value of these programs will be denoted by $\Psi(\bar{x}, p, p^0)$.

Denote *upper and lower Dini derivatives* of the function φ at p in director p^0 by

$$\varphi'_+(p, p^0) = \limsup_{t \downarrow 0} \frac{\varphi(p + tp^0) - \varphi(p)}{t},$$

$$\varphi'_-(p, p^0) = \liminf_{t \downarrow 0} \frac{\varphi(p + tp^0) - \varphi(p)}{t},$$

respectively. The function φ is said to be *first-order directional differentiable at p in direction p^0* (see, for instance, [5]) if $\varphi'_+(p, p^0) = \varphi'_-(p, p^0)$. The common value is denoted by $\varphi'(p, p^0)$ and it is called *first-order directional derivative of φ at p in direction p^0*. Then, we have

$$\varphi'(p, p^0) = \lim_{t \downarrow 0} \frac{\varphi(p + tp^0) - \varphi(p)}{t}.$$

Applying [5, Corollary 1] for the problem $(QP(p))$, we get the following lemma.

Lemma 3.4. *Assume that the problem $(QP(p))$ satisfies (SCQ). Then,*

$$\varphi'_+(p, p^0) \leq \min_{h \in D(\bar{y}, p, p^0)} \left\{ (Q\bar{y} + q)^T h + f(\bar{y}, p^0) \right\} < +\infty.$$

The main result in this section is stated as follows.

Theorem 3.3. *If the problem $(QP(p))$ satisfies (SCQ), (A_3), and (A_4), then φ is first-order directional differentiable at p in every direction $p^0 = (Q^0, q^0, Q^0_1, q^0_1, c^0_1, \ldots, Q^0_m, q^0_m, c^0_m) \in \mathbb{P}$ and*

$$\varphi'(p, p^0) = \min_{\bar{y} \in G(p)} \max_{\lambda \in \Lambda(\bar{y}, p)} \left\{ f(\bar{y}, p^0) + \sum_{i=1}^{m} \lambda_i g_i(\bar{y}, p^0_i) \right\} \tag{3.22}$$

$$= \min_{\bar{y} \in G(p)} \min_{h \in D(\bar{y}, p, p^0)} \left\{ (Q\bar{y} + q)^T h + f(\bar{y}, p^0) \right\}. \tag{3.23}$$

Proof. Suppose that (SCQ), (A_3), and (A_4) hold. From Corollary 1.3 it follows that $G(p) \neq \emptyset$. Let an arbitrary $\bar{y} \in G(p)$. By Lemmas 3.3 and 3.4, we have

$$\begin{aligned}\varphi'_+(p,p^0) &\leq \min_{h\in D(\bar{y},p,p^0)} \Big\{ (Q\bar{y}+q)^T h + f(\bar{y},p^0) \Big\} \\ &= \max_{\lambda\in\Lambda(\bar{y},p)} L(\bar{y},p^0,\lambda).\end{aligned}$$

Since the above inequality holds for any $\bar{y} \in G(p)$,

$$\varphi'_+(p,p^0) \leq \min_{\bar{y}\in G(p)} \max_{\lambda\in\Lambda(\bar{y},p)} L(\bar{y},p^0,\lambda). \tag{3.24}$$

On the other hand, we choose a positive sequence $\{t_k\}$ such that $t_k \to 0$ and

$$\varphi'_-(p,p^0) = \lim_{k\to\infty} \frac{\varphi(p+t_kp^0)-\varphi(p)}{t_k}.$$

By the assumption (SCQ) and Lemma 2.1, the set-valued map $\tilde{p} \mapsto \mathcal{F}(\tilde{p})$ is lower semicontinuous at p. This leads to $\mathcal{F}(p+t_kp^0) \neq \emptyset$ for t_k small enough.

From (A_3) it follows $p \in S$. Since S is open and $p + t_kp^0 \to p$ as $k \to \infty$, there exists $k_0 > 0$ such that $p + t_kp^0 \in S$, for every $k \geq k_0$. By Corollary 1.3, we obtain that $G(p + t_kp^0) \neq \emptyset$, for t_k small enough.

Let a $x^k \in G(p + t_kp^0)$. According to Lemma 3.1, there exists a subsequence $\{x^{k_i}\}$ of $\{x^k\}$ such that $x_{k_i} \to \bar{x} \in G(p)$ as $i \to \infty$. Without loss of generality, we may assume that $\{x^{k_i}\} \equiv \{x^k\}$. For any $\lambda \in \Lambda(\bar{x}, p)$, we have

$$
\begin{aligned}
&\varphi(p+t_kp^0)-\varphi(p)\\
=&f(x^k,p+t_kp^0)-f(\bar{x},p)\\
\geq&L(x^k,p+t_kp^0,\lambda)-L(\bar{x},p,\lambda)\\
=&L(x^k,p,\lambda)-L(\bar{x},p,\lambda)+t_kL(x^k,p^0,\lambda)\\
=&\nabla_xL(\bar{x},p,\lambda)^T(x^k-\bar{x})+\frac{1}{2}(x^k-\bar{x})^T\nabla^2_{xx}L(\bar{x},p,\lambda)(x^k-\bar{x})\\
&+t_kL(x^k,p^0,\lambda)\\
=&\frac{1}{2}(x^k-\bar{x})^T\nabla^2_{xx}L(\bar{x},p,\lambda)(x^k-\bar{x})+t_kL(x^k,p^0,\lambda).
\end{aligned}
$$

Using condition (A_4), we obtain that

$$
\begin{aligned}
\varphi'_-(p,p^0)&=\lim_{k\to\infty}\frac{\varphi(p+t_kp^0)-\varphi(p)}{t_k}\\
=\lim_{k\to\infty}&\left[L(x^k,p^0,\lambda)+\frac{\frac{1}{2}(x^k-\bar{x})^T\big(Q+\sum_{i\in I(\bar{x},p)}\lambda_iQ_i\big)(x^k-\bar{x})}{t_k}\right]\\
\geq&L(\bar{x},p^0,\lambda).
\end{aligned}
$$

Since λ is taken arbitrarily and $\Lambda(\bar{x},p)$ is compact,

$$\varphi'_-(p,p^0)\geq\max_{\lambda\in\Lambda(\bar{y},p)}L(\bar{y},p^0,\lambda). \tag{3.25}$$

Hence

$$\varphi'_-(p,p^0)\geq\min_{\bar{y}\in G(p)}\max_{\lambda\in\Lambda(\bar{y},p)}L(\bar{y},p^0,\lambda). \tag{3.26}$$

Combining (3.24) with (3.26) yields

$$\varphi'(p,p^0)=\min_{\bar{y}\in G(p)}\max_{\lambda\in\Lambda(\bar{y},p)}L(\bar{y},p^0,\lambda).$$

The proof is complete. □

We have the following corollary.

Corollary 3.1. *Consider the problem $(QP(p))$. Assume that one of the assumptions of Propositions 3.1-3.4 is satisfied. Then, φ is first-order directional differentiable at p in every direction $p^0\in\mathbb{P}$ and* (3.22) *holds.*

Proof. By Theorem 3.3 and by Propositions 3.1-3.4, one gets the desired conclusion. □

Note that assumption (A_4) of Theorem 3.3 can be dropped if the feasible region $\mathcal{F}(p)$ is unperturbed. This is shown in the following result.

Theorem 3.4. *If the problem $(QP(p))$ satisfies (A_3), then φ is first-order directional differentiable at p in every direction $p^0 = (Q^0, q^0, 0) \in \mathbb{P}$ and*

$$\varphi'(p, p^0) = \min_{\bar{y} \in G(p)} \left\{ \frac{1}{2} \bar{y}^T Q^0 \bar{y} + (q^0)^T \bar{y} \right\}.$$

Proof. Since $p^0 = (Q^0, q^0, 0) \in \mathbb{P}$, $\mathcal{F}(p) = \mathcal{F}(p + t_k p^0)$. For each $\bar{x} \in G(p)$, we have

$$\begin{aligned} \varphi'_+(p, p^0) &= \limsup_{t \downarrow 0} \tfrac{\varphi(p + tp^0) - \varphi(p)}{t} \\ &\leq \limsup_{t \downarrow 0} \tfrac{f(\bar{x}, p + tp^0) - f(\bar{x}, p)}{t} \\ &= \limsup_{t \downarrow 0} \tfrac{f(\bar{x}, p) + t f(\bar{x}, p^0) - f(\bar{x}, p)}{t} \\ &= f(\bar{x}, p^0). \end{aligned} \tag{3.27}$$

On the other hand, we choose a positive sequence $\{t_k\}$ such that $t_k \to 0$ and

$$\varphi'_-(p, p^0) = \lim_{k \to \infty} \frac{\varphi(p + t_k p^0) - \varphi(p)}{t_k}.$$

For every sequence $\{t_k\}, t_k \downarrow 0$, and for every sequence $\{x^k\}$, $x^k \in G(p + t_k p^0)$ satisfying $x^k \to \bar{x} \in G(p)$, we have

$$\begin{aligned} \varphi(p + t_k p^0) - \varphi(p) &= f(x^k, p + t_k p^0) - f(\bar{x}, p) \\ &= f(x^k, p) - f(\bar{x}, p) + t_k f(x^k, p^0) \\ &\geq t_k f(x^k, p^0). \end{aligned}$$

Hence

$$\varphi'_-(p, p^0) \geq \lim_{k \to \infty} \frac{t_k f(x^k, p^0)}{t_k} \geq f(\bar{x}, p^0) \geq \min_{\bar{y} \in G(p)} f(\bar{y}, p^0). \tag{3.28}$$

Combining (3.27) with (3.28) yields

$$\varphi'(p, p^0) = \min_{\bar{y} \in G(p)} f(\bar{y}, p^0),$$

which proves the theorem. $\square$

3.3. Second-order directional differentiability

The purpose of this section is to derive a set of assumptions which ensures the existence of second-order directional derivative of the optimal value function φ. This property has been studied in [18, Theorem 4.102], [5, Theorem 1], [94, Theorem 4.1] and [72, Theorem 4.1]. The main results in this section differ from those results.

Firstly, we consider the following assumption:

Assumption $(\mathbf{A}_5)$ *For every sequence* $\{t_k\}, t_k \downarrow 0$, *for every sequence* $\{x^k\}$ *satisfying* $x^k \in G(p + t_k p^0)$, $x^k \to \bar{x} \in G(p)$, $h^k := (x^k - \bar{x})/t_k$, *there exists* $\bar{\lambda} \in \Lambda^*(\bar{x}, p)$ *such that*

$$\begin{aligned}&\liminf_{k\to\infty} (h^k, p^0)^T \nabla^2_{(x,p)} L(\bar{x}, p, \bar{\lambda})(h^k, p^0)\\ &\geq \inf_{h \in D^*(\bar{x},p,p^0)} \max_{\lambda \in \Lambda^*(\bar{x},p)} (h, p^0)^T \nabla^2_{(x,p)} L(\bar{x}, p, \lambda)(h, p^0).\end{aligned}$$

Next, we shall show some situations where (A_5) is satisfied.

Proposition 3.5. *Assume that, for every sequence* $\{t_k\}, t_k \downarrow 0$, *for every sequence* $\{x^k\}, x^k \in G(p + t_k p^0)$ *satisfying* $x^k \to \bar{x} \in G(p)$, *the sequence* $\{h^k = (x^k - \bar{x})/t_k\}$ *is bounded. Then,* (A_5) *holds.*

Proof. Without loss of generality, we may assume that $h^k = (x^k - \bar{x})/t_k$ converges to h^0 for some $h^0 \in \mathbb{R}^n$. Dividing both sides of (3.12) by t_k and letting $k \to +\infty$ yields

$$(Q_i\bar{x} + q_i)^T h^0 + g_i(\bar{x}, p^0) \leq 0 \ \forall i \in I(\bar{x}, p),$$

that is, $h^0 \in D(\bar{x}, p, p^0)$. From (3.14),

$$\varphi'(p, p^0) = \lim_{t_k \downarrow 0} \frac{\varphi(p + t_k p^0) - \varphi(p)}{t_k} = (Q\bar{x} + q)^T h^0 + f(\bar{x}, p).$$

This implies $h^0 \in D^*(\bar{x}, p, p^0)$. For any $\lambda \in \Lambda^*(\bar{x}, p)$, we have

$$\begin{aligned} &\liminf_{k\to\infty} (h^k, p^0)^T \nabla^2_{(x,p)} L(\bar{x}, p, \lambda)(h^k, p^0) \\ =&(h^0, p^0)^T \nabla^2_{(x,p)} L(\bar{x}, p, \lambda)(h^0, p^0). \end{aligned}$$

Let

$$\alpha := \inf_{h \in D^*(\bar{x}, p, p^0)} \max_{\lambda \in \Lambda^*(\bar{x}, p)} (h, p^0)^T \nabla^2_{(x,p)} L(\bar{x}, p, \lambda)(h, p^0).$$

Since $D^*(\bar{x}, p, p^0)$ is nonempty, it follows $\alpha < +\infty$.

If α is finite, we assume that

$$\alpha = (\bar{h}, p^0)^T \nabla^2_{(x,p)} L(\bar{x}, p, \bar{\lambda})(\bar{h}, p^0),$$

$\bar{h} \in D^*(\bar{x}, p, p^0)$ and for some $\bar{\lambda} \in \Lambda^*(\bar{x}, p)$. Then,

$$\begin{aligned} &(h^0, p^0)^T \nabla^2_{(x,p)} L(\bar{x}, p, \bar{\lambda})(h^0, p^0) \\ \geq&(\bar{h}, p^0)^T \nabla^2_{(x,p)} L(\bar{x}, p, \bar{\lambda})(\bar{h}, p^0) \\ =&\inf_{h \in D^*(\bar{x}, p, p^0)} \max_{\lambda \in \Lambda^*(\bar{x}, p)} (h, p^0)^T \nabla^2_{(x,p)} L(\bar{x}, p, \lambda)(h, p^0). \end{aligned}$$

Hence

$$\begin{aligned} &\liminf\nolimits_{k\to\infty} (h^k, p^0)^T \nabla^2_{(x,p)} L(\bar{x}, p, \bar{\lambda})(h^k, p^0) \\ \geq\ &\inf\nolimits_{h \in D^*(\bar{x}, p, p^0)} \max\nolimits_{\lambda \in \Lambda^*(\bar{x}, p)} (h, p^0)^T \nabla^2_{(x,p)} L(\bar{x}, p, \lambda)(h, p^0). \end{aligned} \tag{3.29}$$

If $\alpha = -\infty$, we get immediately the inequality (3.29). The proof is complete. □

Remark 3.4. *Both $(SOSC)_{p^0}$ and the assumption of the calmness of global optimal solution mapping (applied for $(QP(p))$) are stronger than (A_5). Indeed, assume that one of these assumptions is satisfied. By Propositions 3.2 and 3.4, we deduce that the sequence $\{(x^k - \bar{x})/t_k\}$ is bounded. From Proposition 3.5 it follows that (A_5) holds.*

For each $h \in D(\bar{x}, p, p^0)$, let

$$I(\bar{x}, h, p, p^0) := \{i \in I(\bar{x}, p) : (Q_i\bar{x} + q_i)^T h + g_i(\bar{x}, p^0) = 0\}.$$

We consider the following proposition.

Proposition 3.6. *Assume that, for every sequence $\{t_k\}, t_k \downarrow 0$, for every sequence $\{x^k\}, x^k \in G(p + t_k p^0)$, $x^k \to \bar{x} \in G(p)$, for every sequence $h^k := (x^k - \bar{x})/t_k$ satisfying $\|h^k\| \to \infty$, the following two conditions is satisfied:*

(b_1) $(Q_i^0\bar{x} + q_i^0)^T(x^k - \bar{x}) \geq 0$ *for every* $i \in I(\bar{x}, p)$*;*

(b_2) $(Q\bar{x} + q)^T v \geq 0 \ \forall v \in \{u \in \mathbb{R}^n : (Q_i\bar{x} + q_i)^T u \leq 0, i \in I(\bar{x}, h^k, p, p^0)\}$ *for every* k *large enough.*

Then, (A_5) *holds.*

Proof. For every $i \in I(\bar{x}, p)$, by the assumption (b_1) and the fact that Q_i^0, $i \in I(\bar{x}, p)$, are positive semidefinite, we obtain that

$$g_i(x^k, p^0) - g_i(\bar{x}, p^0) = \frac{1}{2}(x^k - \bar{x})^T Q_i^0 (x^k - \bar{x}) + (Q_i^0\bar{x} + q_i^0)^T(x^k - \bar{x}) \geq 0.$$

By (3.12), we have

$$(Q_i\bar{x} + q_i)^T(x^k - \bar{x}) + t_k g_i(\bar{x}, p^0) + t_k[g_i(x^k, p^0) - g_i(\bar{x}, p^0)] \leq 0.$$

This is equivalent to

$$(Q_i\bar{x} + q_i)^T(x^k - \bar{x}) + t_k g_i(\bar{x}, p^0) \leq -t_k[g_i(x^k, p^0) - g_i(\bar{x}, p^0)] \leq 0,$$

that is, $(Q_i\bar{x} + q_i)^T \frac{x^k - \bar{x}}{t_k} + g_i(\bar{x}, p^0) \leq 0$. This leads to $h^k \in D(\bar{x}, p, p^0)$.

Let us consider the following problem

$$\min\{(Q\bar{x} + q)^T h + f(\bar{x}, p^0) : h \in D(\bar{x}, p, p^0)\} \tag{LP}$$

The global optimal solution set of (LP) will be denoted by $G(LP)$. By the optimality condition in linear programming (see, for instance, [56]), we deduce that $h \in G(LP)$ is equivalent to

$$(Q\bar{x} + q)^T v \geq 0 \ \forall v \in \{u \in \mathbb{R}^n : (Q_i\bar{x} + q_i)^T u \leq 0, i \in I(\bar{x}, h, p, p^0)\}.$$

Using the assumption (b_2), we have $h^k = (x^k - \bar{x})/t_k \in G(LP)$, that is,

$$(Q\bar{x}+q)^T h^k + f(\bar{x},p^0) = \min_{h\in D(\bar{x},p,p^0)} \big\{(Q\bar{x}+q)^T h + f(\bar{x},p^0)\big\}.$$

This implies $h^k \in D^*(\bar{x},p,p^0)$.

Let

$$\beta := \inf_{h\in D^*(\bar{x},p,p^0)} \max_{\lambda\in\Lambda^*(\bar{x},p)} (h,p^0)^T \nabla^2_{(x,p)} L(\bar{x},p,\lambda)(h,p^0).$$

Since $D^*(\bar{x},p,p^0)$ is nonempty, it follows $\beta < +\infty$.

If β is finite, we assume that $\beta = (\bar{h},p^0)^T\nabla^2_{(x,p)}L(\bar{x},p,\bar{\lambda})(\bar{h},p^0)$, for some $\bar{h}\in D^*(\bar{x},p,p^0)$ and for some $\bar{\lambda}\in\Lambda^*(\bar{x},p)$. Then, for each k, we have

$$\begin{aligned}
&(h^k,p^0)^T\nabla^2_{(x,p)}L(\bar{x},p,\bar{\lambda})(h^k,p^0)\\
\geq&(\bar{h},p^0)^T\nabla^2_{(x,p)}L(\bar{x},p,\bar{\lambda})(\bar{h},p^0)\\
=&\inf_{h\in D^*(\bar{x},p,p^0)} \max_{\lambda\in\Lambda^*(\bar{x},p)} (h,p^0)^T \nabla^2_{(x,p)} L(\bar{x},p,\lambda)(h,p^0).
\end{aligned}$$

Hence

$$\begin{aligned}
&\liminf_{k\to\infty}(h^k,p^0)^T\nabla^2_{(x,p)}L(\bar{x},p,\bar{\lambda})(h^k,p^0)\\
\geq\ &\inf_{h\in D^*(\bar{x},p,p^0)}\max_{\lambda\in\Lambda^*(\bar{x},p)}(h,p^0)^T\nabla^2_{(x,p)}L(\bar{x},p,\lambda)(h,p^0).
\end{aligned}\tag{3.30}$$

If $\beta = -\infty$, we get immediately the inequality (3.30). The proof is complete. $\square$

Let

$$G(p,p^0) := \{\bar{y}\in G(p) : \varphi'(p,p^0) = \Psi(\bar{y},p,p^0)\},$$

where

$$\Psi(\bar{y},p,p^0) = \min_{h\in D(\bar{y},p,p^0)} \big\{(Q\bar{y}+q)^T h + f(\bar{y},p^0)\big\}.$$

Denote

$$\varphi''_+(p,p^0) = \limsup_{t\downarrow 0} \frac{\varphi(p+tp^0)-\varphi(p)-t\varphi'(p,p^0)}{\frac{t^2}{2}},$$

$$\varphi''_-(p, p^0) = \liminf_{t \downarrow 0} \frac{\varphi(p + tp^0) - \varphi(p) - t\varphi'(p, p^0)}{\frac{t^2}{2}}.$$

The function φ is said to be *second-order directional differentiable* at p in direction p^0 if $\varphi''_+(p, p^0) = \varphi''_-(p, p^0)$. The common value is denoted by $\varphi''(p, p^0)$ and it is called *second-order directional derivative* of φ at p in direction p^0. Then,

$$\varphi''(p, p^0) = \lim_{t \downarrow 0} \frac{\varphi(p + tp^0) - \varphi(p) - t\varphi'(p, p^0)}{\frac{t^2}{2}}.$$

Let

$$K(\bar{x}, h, p, p^0) := \{v \in \mathbb{R}^n : (Q_i\bar{x} + q_i)^T v + (h, p^0)^T \nabla^2 g_i(\bar{x}, p)(h, p^0) \leq 0, \\ i \in I(\bar{x}, h, p, p^0))\}.$$

Applying [5, Lemma 4] for $(QP(p))$, we get the following lemma.

Lemma 3.5. *Consider the problem $(QP(p))$ and $p^0 \in \mathbb{P}$. If (SCQ) holds, then for each $h \in D^*(\bar{x}, p, p^0)$, the following two extrema are attained and equal, i.e.,*

$$\min_{v \in K(\bar{x}, h, p, p^0)} \Big[2(Q\bar{x} + q)^T v + (h, p^0)^T \nabla^2 f(\bar{x}, p)(h, p^0) \Big] \\ = \max_{\lambda \in \Lambda^*(\bar{x}, p)} (h, p^0)^T \nabla^2_{(x,p)} L(\bar{x}, p, \lambda)(h, p^0).$$

The common value of the above programs will be denoted by $\Phi(\bar{x}, h, p, p^0)$.

The main result in this section is presented as follows.

Theorem 3.5. *Consider the problem $(QP(p))$. If assumptions (SCQ) and (A_3)–(A_5) are satisfied, then φ is second-order directional differentiable at p in direction $p^0 = (Q^0, q^0, Q^0_1, q^0_1, c^0_1, \ldots, Q^0_m, q^0_m, c^0_m) \in \mathbb{P}$ and*

$$\varphi''(p, p^0) = \min_{\bar{x} \in G(p, p^0)} \inf_{h \in D^*(\bar{x}, p, p^0)} \max_{\lambda \in \Lambda^*(\bar{x}, p)} \Big\{ h^T \Big(Q + \sum_{i \in I(\bar{x}, p)} \lambda_i Q_i \Big) h + \\ + 2 \Big(Q^0 \bar{x} + q^0 + \sum_{i \in I(\bar{x}, p)} \lambda_i (Q^0_i \bar{x} + q^0_i) \Big)^T h \Big\}.$$

$$(3.31)$$

Proof. Let any $\bar{y} \in G(p, p^0)$ and $\bar{h} \in D^*(\bar{y}, p, p^0)$. According to Theorem 2, we obtain

$$\varphi'(p, p^0) = (Q\bar{y} + q)^T \bar{h} + f(\bar{y}, p^0).$$

Using [5, Proposition 1], we get

$$\varphi''_+(p, p^0) \le \min_{v \in K(\bar{y}, \bar{h}, p, p^0)} \Big[2(Q\bar{y} + q)^T v + (\bar{h}, p^0)^T \nabla^2 f(\bar{y}, p)(\bar{h}, p^0) \Big].$$

From Lemma 3.5 it follows immediately that $\varphi''_+(p, p^0) \le \Phi(\bar{y}, \bar{h}, p, p^0)$. Since $\bar{y} \in G(p, p^0)$ and since $\bar{h} \in D^*(\bar{y}, p, p^0)$ are taken arbitrarily, we have

$$\varphi''_+(p, p^0) \le \min_{\bar{y} \in G(p, p^0)} \inf_{h \in D^*(\bar{y}, p, p^0)} \Phi(\bar{y}, h, p, p^0). \tag{3.32}$$

Suppose that $\varphi''_-(p, p^0)$ be attained on the sequence $t_k \downarrow 0$. We let $x^k \in G(p + t_k p^0)$ such that $x^k \to \bar{x} \in G(p)$. From(3.25) it follows that

$$\varphi'(p, p^0) = \varphi'_-(p, p^0) \ge \Psi(\bar{x}, p, p^0).$$

Since $\varphi'(p, p^0) = \min_{\bar{y} \in G(p)} \Psi(\bar{y}, p, p^0) \le \Psi(\bar{x}, p, p^0)$, we have

$$\varphi'(p, p^0) = \Psi(\bar{x}, p, p^0).$$

Hence

$$\bar{x} \in G(p, p^0). \tag{3.33}$$

Take any $\lambda \in \Lambda^*(\bar{x}, p)$. Since $\bar{x} \in G(p)$, $\nabla_x L(\bar{x}, p, \lambda) = 0$. Hence

$$\begin{aligned}
& \varphi(p + t_k p^0) - \varphi(p) - t_k \varphi'(p, p^0) \\
\ge \; & L(x^k, p + t_k p^0, \lambda) - L(\bar{x}, p, \lambda) \\
& - t_k \nabla_p L(\bar{x}, p, \lambda)^T p^0 - t_k \nabla_x L(\bar{x}, p, \lambda)^T (x^k - \bar{x}) \\
= \; & \tfrac{1}{2}(x^k - \bar{x}, t_k p^0)^T \nabla^2_{(x,p)} L(\bar{x}, p, \lambda)(x^k - \bar{x}, t_k p^0).
\end{aligned} \tag{3.34}$$

Using (A_5), there exists $\bar{\lambda} \in \Lambda^*(\bar{x}, p)$ such that

$$\begin{aligned}
\varphi''_-(p, p^0) &= \lim_{k\to\infty} \frac{\varphi(p + t_k p^0) - \varphi(p) - t_k \varphi'(p, p^0)}{\frac{t_k^2}{2}} \\
&\geq \lim_{k\to\infty} \left(\frac{x^k - \bar{x}}{t_k}, p^0\right)^T \nabla^2_{(x,p)} L(\bar{x}, p, \bar{\lambda}) \left(\frac{x^k - \bar{x}}{t_k}, p^0\right) \\
&\geq \inf_{h\in D^*(\bar{x},p,p^0)} \max_{\lambda\in\Lambda^*(\bar{x},p)} (h, p^0)^T \nabla^2_{(x,p)} L(\bar{x}, p, \lambda)(h, p^0) \\
&= \inf_{h\in D^*(\bar{x},p,p^0)} \Phi(\bar{x}, h, p, p^0).
\end{aligned}$$

From (3.33) it follows that

$$\varphi''_-(p, p^0) \geq \min_{\bar{y}\in G(p,p^0)} \inf_{h\in D^*(\bar{y},p,p^0)} \Phi(\bar{y}, h, p, p^0). \tag{3.35}$$

By (3.32) and (3.35), we get the desired conclusion. □

We have the following corollary.

Corollary 3.2. *Consider the problem* $(QP(p))$ *and* $p^0 \in \mathbb{P}$. *Assume that (SCQ),* (A_3), *and at least one of the following conditions is satisfied:*
i) (A_4) *and the assumptions of Proposition* (3.6) *hold;*
ii) $(SOSC)_{p^0}$ *holds at* $\bar{x} \in G(p)$;
iii) $G(\cdot)$ *is calm at* $(p, \bar{x}) \in \mathbb{P} \times G(p)$.
Then, φ *is second-order directional differentiable at* p *in direction* p^0 *and* (3.31) *holds.*

Proof. By Theorem 3.5, Propositions 3.1–3.4, Proposition 3.6 and Remark 3.4, we have the desired conclusion. □

The following result gives a sufficient condition for second-order directional differentiability of the optimal value function.

Theorem 3.6. *Assume that the problem* $(QP(p))$ *satisfies* (SCQ) *and* (A_3), *and* φ *is first-order directional differentiable at* p *in every direction*

$p^0 \in \mathbb{P}$. Assume that there exists $\bar{\lambda} \in \Lambda(\bar{x}, p)$ and, for every $t \downarrow 0$, $x_t \in G(p + tp^0), x_t \to \bar{x} \in G(p)$ such that

$$\lim_{t \downarrow 0} (h_t, p^0)^T \nabla^2_{(x,p)} L(\bar{x}, p, \bar{\lambda})(h_t, p^0)$$

exists, where $h_t = (x_t - \bar{x})/t$, and

$$\lim_{t \downarrow 0} \frac{\bar{\lambda}_i g_i(x_t, p + tp^0)}{t^2} = 0.$$

Then φ is second-order directional differentiable at p in direction p^0 and

$$\varphi''(p, p^0) = \lim_{t \downarrow 0} \Bigg[h_t^T \Big(Q + \sum_{i \in I(\bar{x}, p)} \bar{\lambda}_i Q_i \Big) h_t + \\ + 2 \Big(Q^0 \bar{x} + q^0 + \sum_{i \in I(\bar{x}, p)} \bar{\lambda}_i (Q_i^0 \bar{x} + q_i^0) \Big)^T h_t \Bigg].$$

Proof. For every $t > 0$ small enough, let $x_t \in G(p + tp^0), x_t \to \bar{x} \in G(p)$ and $h_t = (x_t - \bar{x})/t$. We have

$$\begin{aligned} &\varphi(p + tp^0) - \varphi(p) - t\varphi'(p, p^0) \\ =&L(x_t, p + tp^0, \bar{\lambda}) - \sum_{i \in I(\bar{x}, p)} \lambda_i g_i(x_t, p + tp^0) \\ &- L(\bar{x}, p, \bar{\lambda}) - t\nabla_p L(\bar{x}, p, \bar{\lambda})^T p^0 - t\nabla_x L(\bar{x}, p, \bar{\lambda})^T (x_t - \bar{x}) \\ =&\frac{1}{2}(x_t - \bar{x}, tp^0)^T \nabla^2_{(x,p)} L(\bar{x}, p, \bar{\lambda})(x_t - \bar{x}, tp^0) - \sum_{i \in I(\bar{x}, p)} \bar{\lambda}_i g_i(x_t, p + tp^0). \end{aligned}$$

Hence

$$\begin{aligned} \varphi''(p, p^0) =& \lim_{t \downarrow 0} \frac{\varphi(p + tp^0) - \varphi(p) - t\varphi'(p, p^0)}{\frac{t^2}{2}} \\ =& \lim_{t \downarrow 0} \Bigg[(h_t, p^0)^T \nabla^2_{(x,p)} L(\bar{x}, p, \bar{\lambda})(h_t, p^0) + \\ & - 2 \sum_{i \in I(\bar{x}, p)} \frac{\bar{\lambda}_i g_i(x_t, p + tp^0)}{t^2} \Bigg] \\ =& \lim_{t \downarrow 0} (h_t, p^0)^T \nabla^2_{(x,p)} L(\bar{x}, p, \bar{\lambda})(h_t, p^0), \end{aligned}$$

since $\lim_{t \downarrow 0} \frac{\bar{\lambda}_i g_i(x_t, p + tp^0)}{t^2} = 0$. This ends the proof. $\square$

The following two examples illustrate some applications of the above theorems. We also show that Theorem 1 in [5], Theorems 3.1 and 4.1 in [72] could not be applied for these problems.

Example 3.2. *We consider the problem* $(QP(p))$ *with*

$$Q = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad q = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

$$Q_1 = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}, \quad q_1 = \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \quad c_1 = 0,$$

$$Q_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad q_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \quad c_2 = 0,$$

$p = (Q, q, Q_1, q_1, c_1, Q_2, q_2, c_2) \in \mathbb{P}$, *and*

$$\mathbb{P} = \mathbb{R}^{2\times 2}_S \times \mathbb{R}^2 \times (\mathbb{R}^{2\times 2}_{S^+} \times \mathbb{R}^2 \times \mathbb{R}) \times (\mathbb{R}^{2\times 2}_{S^+} \times \mathbb{R}^2 \times \mathbb{R}).$$

This problem can be rewritten as follows

$$\min\left\{ f(x,p) = \frac{1}{2}(x_1^2 - x_2^2), x \in \mathcal{F}(p) \right\},$$

where $\mathcal{F}(p) = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 - x_2 \le 0; -x_1 + x_2 \le 0\}$.

Since $f(x,p) = \frac{1}{2}(x_1^2 - x_2^2) \ge 0$ *for all* $x \in \mathcal{F}(p)$, *we have*

$$G(p) = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 = x_2, 0 \le x_1 \le 1\}.$$

Let

$$Q^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad q^0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

$$Q_1^0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad q_1^0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad c_1^0 = 1,$$

$$Q_2^0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad q_2^0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad c_2^0 = 0,$$

and $p^0 = (Q^0, q^0, Q_1^0, q_1^0, c_1^0, Q_2^0, q_2^0, c_2^0) \in \mathbb{P}$.

Firstly, it is easy to verify that (SCQ) holds and $\mathcal{F}(p)$ *is bounded. Thus* (A_3) *holds. Using Theorem 3.1, we get* φ *is continuous at* p.

Next, we show that φ *is first-order directional differentiable at* p *in every direction* p^0. *One gets*

$$\mathcal{F}(p+tp^0) = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 - x_2 + t \le 0; -x_1 + x_2 \le 0\}$$

and

$$f(x, p+tp^0) = \frac{1}{2}(1+t)(x_1^2 - x_2^2) \ge 0, \forall x \in \mathcal{F}(p+tp^0).$$

For t *small enough, we obtain*

$$G(p+tp^0) = \left\{(x_1, x_1) \in \mathbb{R}^2 : \frac{1-\sqrt{1-4t}}{2} \le x_1 \le \frac{1+\sqrt{1-4t}}{2}\right\}.$$

For each $x_1 \in [0,1]$, *we get* $\Lambda((x_1, x_1), p) = \{(0, x_1)\}$.

For any $t_k \downarrow 0$, *for any* $x^k = (x_1^k, x_1^k) \in G(p + t_k p^0)$ *satisfying* $x^k \to \bar{x} = (\bar{x}_1, \bar{x}_1)$, *we have*

$$\begin{aligned}
&\liminf_{k\to+\infty} \frac{(x^k - \bar{x})^T \left(Q + \sum_{i\in I(\bar{x},p)} \lambda_i Q_i\right)(x^k - \bar{x})}{t_k} \\
=&\liminf_{k\to+\infty} \frac{(x^k - \bar{x})^T Q (x^k - \bar{x})}{t_k} \\
=&\liminf_{k\to+\infty} [(x_1^k - \bar{x}_1)^2 - (x_1^k - \bar{x}_1)^2] \\
=&0.
\end{aligned}$$

Hence (A_4) *holds. Using Theorem 3.3, it implies that* φ *is first-order directional differentiable at* p *in every direction* p^0 *and*

$$\begin{aligned}
\varphi'(p, p^0) &= \min_{\bar{y}\in G(p)} \max_{\lambda\in\Lambda(\bar{y},p)} \left[f(\bar{y}, p^0) + \sum_{i=1}^{2} \lambda_i g_i(\bar{y}, p_i^0)\right] \\
&= \min_{\bar{y}\in G(p)} f(\bar{y}, p^0) \\
&= \min_{\bar{y}=(y_1,y_1)\in G(p)} \frac{1}{2}(y_1^2 - y_1^2) \\
&=0.
\end{aligned}$$

We next show that φ is second-order directional differentiable at p in every direction p^0. Since $Q_i^0\bar{x} + q_i^0 = 0$ for every $i \in \{1, 2\}$, it follows $(Q_i^0\bar{x} + q_i^0)^T(x^k - \bar{x}) \geq 0$. Hence the condition (b_1) of Proposition 3.6 is satisfied. To check the condition (b_2) of Proposition 3.6, we consider the point $\bar{x} = (x_1, x_1)$. If $x_1 = 0$, then $(Q\bar{x} + q)^T v = 0$ for every $v \in \mathbb{R}^2$. If $0 < x_1 \leq 1$, then

$$(Q\bar{x} + q)^T v = \bar{x}_1(v_1 - v_2) \geq 0$$

for every $v = (v_1, v_2)$ satisfying $-v_1 + v_2 \leq 0$. Hence the condition (b_2) of Proposition 3.6 is satisfied. Hence (A_5) holds. Using Theorem 3.5 and Proposition 3.6, we deduce that φ is second-order directional differentiable at p in every direction p^0.

For $\bar{x} = (x_1, x_1) \in G(p), 0 \leq x_1 \leq 1$, we obtain that

$$D(\bar{x}, p, p^0) = \{h = (h_1, h_2) : 2x_1h_1 - h_2 + 1 \leq 0; -h_1 + h_2 \leq 0\}$$

and

$$(Q\bar{x} + q)^T h = x_1(h_1 - h_2) \geq 0 \quad \forall h = (h_1, h_2) \in D(\bar{x}, p, p^0).$$

Hence

$$\min_{h \in D(\bar{x}, p, p^0)} (Q\bar{x} + q)^T h = 0$$

and

$$D^*(\bar{x}, p, p^0) = \{(h_1, h_1) \in \mathbb{R}^2 : (2x_1 - 1)h_1 + 1 \leq 0\}.$$

Therefore

$$\begin{aligned}
\varphi''(p, p^0) &= \min_{\bar{x} \in G(p, p^0)} \min_{h \in D^*(\bar{x}, p, p^0)} \Phi(\bar{x}, h, p, p^0) \\
&= \min_{\bar{x} \in G(p, p^0)} \min_{h \in D^*(\bar{x}, p, p^0)} (h_1 - h_2)(h_1 + h_2 + x_1) \\
&= 0.
\end{aligned}$$

Finally, we show that both $(SOSC)_{p^0}$ and the assumption of the calmness of the global optimal solution mapping are not satisfied. Indeed, *let* $x^k = (\frac{1}{2} + t_k^{\frac{1}{2}}, \frac{1}{2} + t_k^{\frac{1}{2}}) \in G(p + t_k p^0)$. *Then, we get*

$$x^k \to \bar{x} = \left(\frac{1}{2}, \frac{1}{2}\right), h^k = \frac{x^k - \bar{x}}{t_k} = \left(\frac{t_k^{\frac{1}{2}}}{t_k}, \frac{t_k^{\frac{1}{2}}}{t_k}\right) = \left(\frac{1}{t_k^{\frac{1}{2}}}, \frac{1}{t_k^{\frac{1}{2}}}\right)$$

and

$$\|h^k\| \to +\infty \text{ as } k \to +\infty.$$

By Propositions 3.2 and 3.4, we have the desired conclusion. Therefore Theorem 1 in [5], Theorems 3.1 and 4.1 in [72] can not be applied for $(QP(p))$ *in this example.*

Example 3.3. *We consider the problem* $(QP(p))$ *with*

$$Q = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}, \quad q = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

$$Q_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad q_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad c_1 = -\frac{1}{2},$$

and $p = (Q, q, Q_1, q_1, c_1) \in \mathbb{P} = \mathbb{R}_S^{2\times 2} \times \mathbb{R}^2 \times (\mathbb{R}_{S^+}^{2\times 2} \times \mathbb{R}^2 \times \mathbb{R})$.

This problem can be rewritten as follows

$$\min \left\{ f(x,p) = -\frac{1}{2}x_1^2 + x_2, x \in \mathcal{F}(p) \right\},$$

where $\mathcal{F}(p) = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 \leq 1\}$.

Let

$$Q^0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad q^0 = \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \quad Q_1^0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad q_1^0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad c_1^0 = 0$$

and $p^0 = (Q^0, q^0, Q_1^0, q_1^0, c_1^0) = (Q^0, q^0, 0) \in \mathbb{P}$.

We check that $G(p) = \{\bar{x} = (0, -1)\}$, *and* $\Lambda(\bar{x}, p) = \{\lambda_1 = 1\}$.

Firstly, it is easy to verify that (SCQ) *holds and* $\mathcal{F}(p)$ *is bounded. Thus* (A_3) *holds. Using Theorem 3.1, one implies that* φ *is continuous at* p.

Next, we show that φ is first-order directional differentiable at p in every direction p^0. Indeed, one gets

$$G(p+tp^0) = \left\{(\sqrt{2t-t^2}, t-1), -(\sqrt{2t-t^2}, t-1)\right\}.$$

Using Theorem 3.4, it implies that φ is first-order directional differentiable at p in every direction p^0 and

$$\varphi'(p,p^0) = \min_{\bar{y}\in G(p)} f(\bar{y},p^0) = f(\bar{x},p^0) = -1.$$

Let $x_t = (\sqrt{2t-t^2}, t-1)$. Then,

$$h_t = \frac{x_t - \bar{x}}{t} = \left(\sqrt{\frac{2}{t}-1}, 1\right).$$

We have $g_1(x_t, p+tp^0) = 0$ for every $t > 0$ and

$$\begin{aligned}
&\lim_{t\downarrow 0}(h_t,p^0)^T\nabla^2_{(x,p)}L(\bar{x},p,\lambda)(h_t,p^0)\\
=&\lim_{t\downarrow 0}\left[h_t^T(Q+\lambda_1 Q_1)h_t + 2\left(Q^0\bar{x}+q^0+\lambda_1(Q_1^0\bar{x}+q_1^0)\right)^T h_t\right]\\
=&\lim_{t\downarrow 0}(1-2)\\
=&-1.
\end{aligned}$$

Using Theorem 3.6, we deduce that φ is second-order directional differentiable at p in direction p^0 and

$$\varphi''(p,p^0) = \lim_{t\downarrow 0}\left[h_t^T(Q+\lambda_1 Q_1)h_t+2\left(Q^0\bar{x}+q^0+\lambda_1(Q_1^0\bar{x}+q_1^0)\right)^T h_t\right] = -1.$$

On the other hand, we have

$$D^*(\bar{x},p,p^0) = \{h = (h_1, 0) : h_1 \in \mathbb{R}\}$$

and

$$(h,p^0)^T\nabla^2_{(x,p)}L(\bar{x},p,\lambda)(h,p^0 = h_2^2 - 2h_2 = 0 \quad \forall h \in D^*(\bar{x},p,p^0).$$

Hence

$$\inf_{h\in D^*(\bar{x},p,p^0)} \max_{\lambda\in\Lambda^*(\bar{x},p)} (h,p^0)^T \nabla^2_{(x,p)} L(\bar{x},p,\lambda)(h,p^0) = 0. \tag{3.36}$$

For every sequence $\{t_k\}, t_k \downarrow 0$, *for every sequence* $\{x^k\}$, $x^k \in G(p+t_kp^0)$, $x^k \to \bar{x} \in G(p)$, $h^k := (x^k - \bar{x})/t_k$, *we have*

$$\liminf_{k\to\infty} (h^k,p^0)^T \nabla^2_{(x,p)} L(\bar{x},p,\lambda)(h^k,p^0) = -1. \tag{3.37}$$

From (3.36) *and* (3.37) *it follows that* (A_5) *is not satisfied. Therefore Theorem 3.5 can not be applied for* $QP(p)$.

Finally, from the above one has

$$h^k = \frac{x^k - \bar{x}}{t_k} = \Big(\pm\sqrt{-1+\frac{2}{t_k}}, 1\Big),$$

and $\|h^k\| \to +\infty$ *as* $k \to +\infty$. *By Propositions 3.2 and 3.4, both condition* $(SOSC)_{p^0}$ *and the assumption of the calmness of global optimal solution mapping are not satisfied. Therefore Theorem 1 in [5], Theorems 3.1 and 4.1 in [72] could not be applied for* $(QP(p))$ *in this example.*

3.4. Conclusions

This chapter has presented some results on the continuity and directional differentiability of the optimal value function of $(QP(p))$ (Theorems 3.3–3.6) under weaker assumptions in comparison with the results which are implied from general theory. In some cases, (A_3) is weaker than the uniformly compactness of the constraint set mapping in [36, Theorem 3.3] applied for $(QP(p))$; both (A_4) and (A_5) are weaker than $(SOSC)_{p^0}$ in [5, Theorem 1] and the assumption of the calmness of global optimal solution mapping in [72, Theorems 3.1 and 4.1] applied for $(QP(p))$; in some cases, (A_4) is also weaker than $(H3)$ in [71, Theorem 4.1] applied for $(QP(p))$. Applying Theorems 3.1 and 3.3 for LCQP problems, we get some results which have been investigated in [56, 98, 100].

Chapter 4

Stability for extended trust region subproblems

This chapter devotes detailed discussion to the stability for parametric extended trust region subproblem (ETRS), which is a class of QCQP problems. In Section 4.1, the ETRS is stated. Some stability results for ETRS are established in Section 4.2. In Sections 4.3, we calculate and estimate the Fréchet and Mordukhovich coderivatives of the normal cone mapping related to the parametric ETRS. We also use the obtained results and the Mordukhovich criterion (see [73, Theorem 4.10]) for the local Lipschitz-like property of multifunctions to investigate Lipschitzian stability of ETRS with respect to the linear perturbations.

The material of this chapter is taken from [76, 78, 79].

4.1. Problem statement

In this section, we concern to parametric ETRS as follows

$$\begin{aligned} &\min\ f(x, Q, q) := \tfrac{1}{2}x^T Qx + q^T x \\ &\text{s.t. } x \in \mathbb{R}^n : x^T Dx \le r,\ Ax + b \le 0, \end{aligned} \qquad (ET_m(w))$$

where $Q, D \in \mathbb{R}^{n\times n}$ are symmetric, D is positive definite, $q \in \mathbb{R}^n$, $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$, $r > 0$ and $w := (Q, q, D, r, A, b)$. Denote

$$\mathbb{W} := \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n} \times \mathbb{R}^n \times (\mathbb{R}_+ \setminus \{0\}) \times \mathbb{R}^{m\times n} \times \mathbb{R}^m.$$

Model problems of this type are widespread in real-world applications such as: nonlinear programming problems with linear inequality constraints, nonlinear optimization problems with discrete variables [21, 82] and robust optimization problems under matrix norm or polyhedral uncertainty, optimal control and system theory (see [48, 92]).

Without the first constraint, ETRS reduces the LCQP problem. A survey on stability for parametric LCQP was investigated by Lee et al. [56].

The special case of ETRS, where D is the unit matrix and $m = 0$, is the well-known TRS, which plays an important role in trust region methods for nonlinear optimization (see [24, 67, 70]). The stability for parametric TRSs has been concerned by many authors. Lee et al. [61] obtained necessary and sufficient conditions for the upper/lower semicontinuity of the stationary solution map and the global optimal solution map, explicit formulas for computing the directional derivative and the Fréchet derivative of the optimal value function. The local Lipschitz-like property of the stationary solution map was characterized in [62, 85].

ETRS is a generalization of TRS and LCQP. It is a common agreement that linear and pure quadratic forms are relatively easy but their combination is not. Recently, some topics related to ETRS have been investigated:

(i) Beck and Eldar [11], Jeyakumar and Li [48] showed the necessary and sufficient optimality conditions for the global optimality of ETRS;

(ii) Some methods to find the global optimal solution for the general problem of ETRS have been proposed (see [22, 92, 93]);

(iii) ETRS with a linear constraint ($m = 1$) has been mentioned in literatures (see [22, 76, 78, 92]).

Stability for parametric ETRSs plays an important role because they can be used for analyzing algorithms for solving this problem. Since ETRSs form a subclass of nonlinear optimization problems, many interesting results on stability for parametric nonlinear optimization (see for instance, [53,54,89]) and QCQP problems (Chapters 2–3) can be applied to parametric ETRSs. As far as we know, up to now, there have been a few studies which approach directly the stability for parametric ETRS.

In next sections, we use the special structure of ETRS (the objective function is quadratic and the trust region intersects an ellipsoid solid with many linear inequality constraints) to obtain deeper and sharper results on stability of this problem.

4.2. Some stability results for parametric ETRS

In this section, we investigate in details continuity of the stationary solution map and the optimal value function to parametric ETRS with several illustrated examples. The obtained results herein develop and complement the published ones in [56,61]. The approach adopted herein is quite different from that used in [62, 85].

4.2.1. Continuity of the stationary solution map

The upper semicontinuity of the stationary solution map follows from Theorem 2.4. In this subsection, we investigate the lower semicontinuity of $S(\cdot)$.

Let

$$h_0(x, w) := x^T Dx - r^2,\ h_i(x, w) := A_i x + b_i,\ i = 1, \ldots, m. \qquad (4.1)$$

The KKT pair of $(ET_m(w))$ is rewritten (x, λ, μ), where $x \in S(w)$, $(\lambda, \mu) \in \mathbb{R} \times \mathbb{R}^m$.

The following lemma is useful for proving the main results

Lemma 4.1. *If the problem $(ET_m(w))$ does not satisfy (SCQ) then there exists $b^k \to b$ such that, for each k, the set $\mathcal{F}(Q, q, D, r, A, b^k)$ is empty.*

Proof. Let $b^k \downarrow b$ and fix any $x \in \mathbb{R}^n$. Since $(ET_m(\bar{w}))$ does not satisfy (SCQ), we obtain that either $h_0(x, w) = 0$ or $h_i(x, w) = 0$ for some $i \in \{1, \ldots, m\}$.

If there exists an index $i \in \{1, \ldots, m\}$ such that $h_i(x, w) = 0$, then $A_i x + b_i^k > A_i x + b_i = 0$. Hence $x \notin \mathcal{F}(Q, q, D, r, A, b^k)$.

If $h_0(x, w) = 0$ and $h_i(x, w) < 0$ for every $i \in \{1, \ldots, m\}$, then there exists a sequence $\{x^s\}$ such that $x^s \to x$ and $h_0(x^s, w) < 0$. Thus for s large enough, $Ax^s + b < 0$. It implies that $(ET_m(w))$ satisfies (SCQ), contrary to the assumption. The proof is complete. $\square$

The necessary condition for the lower semicontinuity of the multifunction $S(\bar{Q}, ., \bar{D}, \bar{r}, \bar{A}, .)$ is characterized in the following theorem.

Theorem 4.1. *Consider $(ET_m(w))$ and $\bar{w} = (\bar{Q}, \bar{q}, \bar{D}, \bar{r}, \bar{A}, \bar{b}) \in \mathbb{W}$. If $\bar{A}$ has full rank and $S(\bar{Q}, ., \bar{D}, \bar{r}, \bar{A}, .)$ is lower semicontinuous at $(\bar{q}, \bar{b})$, then $(ET_m(\bar{w}))$ satisfies (SCQ) and $S(\bar{w})$ is a nonempty set which contains at most 2^m points.*

Proof. We first show that $(ET_m(\bar{w}))$ satisfies (SCQ). Indeed, if $(ET_m(\bar{w}))$ does not satisfy (SCQ), there exists $b^k \to \bar{b}$ such that, for each k, $\mathcal{F}(\bar{D}, \bar{r}, \bar{A}, b^k)$ is empty by Lemma 4.1. Then $S(\bar{Q}, \bar{q}, \bar{r}, \bar{A}, b^k) = \emptyset$ for all $k \in \mathbb{N}$ and $S(\bar{Q}, ., \bar{D}, \bar{r}, \bar{A}, .)$ cannot be lower semicontinuous at $(\bar{q}, \bar{b})$. This contradicts the assumption. Thus $(ET_m(\bar{w}))$ satisfies (SCQ).

For each $\emptyset \neq S \subset \{1, \ldots, m\}$ and for each $t \in \mathbb{R}$, let

$$\mathcal{A}_S := \begin{pmatrix} \bar{Q} & \bar{A}_S \\ & \\ \bar{A}_S^T & 0 \end{pmatrix} \text{ and } \mathcal{A}_{0S}(t) := \begin{pmatrix} \bar{Q} + t\bar{D} & \bar{A}_S \\ & \\ \bar{A}_S^T & 0 \end{pmatrix},$$

where $\bar{A}_S = (\bar{A}_{ij})_{i \in S,\, j=1,\ldots,m}$. If $S = \emptyset$ then we let $\mathcal{A}_S = \bar{Q}$ and let $\mathcal{A}_{0S}(t) = \bar{Q} + t\bar{D}$.

Since $\bar{D}$ is positive definite, there exists an orthogonal matrix C such that $C^{-1}\bar{D}C = I$, where I denotes the $n \times n$ unit matrix. We denote the set of eigenvalues of $C^{-1}\bar{Q}C$ by T.

For each $S \subset \{1, \ldots, m\}$, let:

$$\mathcal{P}_S := \Bigg\{ (u, v) \in \mathbb{R}^n \times \mathbb{R}^m : \begin{pmatrix} u \\ v_S \end{pmatrix} = \mathcal{A}_S \begin{pmatrix} x \\ \mu_S \end{pmatrix} \text{ for some } (x, \mu) \in \mathbb{R}^n \times \mathbb{R}^m \Bigg\};$$

$$\mathcal{P}_{0S}(t) := \Bigg\{ (u, v) \in \mathbb{R}^n \times \mathbb{R}^m : \begin{pmatrix} u \\ v_S \end{pmatrix} = \mathcal{A}_{0S}(t) \begin{pmatrix} x \\ \mu_S \end{pmatrix} \text{ for some } (x, \mu) \in \mathbb{R}^n \times \mathbb{R}^m \Bigg\};$$

$$\begin{aligned} \mathcal{P} := & \cup \big\{ \mathcal{P}_S : S \subset \{1, \ldots, m\}, det\mathcal{A}_S = 0 \big\} \\ & \cup \big\{ \mathcal{P}_{0S}(t) : S \subset \{1, \ldots, m\}, t \in T \big\}. \end{aligned}$$

For each $S \subset \{1, \ldots, m\}$, if $det\mathcal{A}_S = 0$, then $\mathcal{P}_S$ is a proper linear subspace of $\mathbb{R}^n \times \mathbb{R}^m$. If $t \in T$ then $det\mathcal{A}_S(t) = 0$; hence $\mathcal{P}_{0S}(t)$ is also a proper linear subspace of $\mathbb{R}^n \times \mathbb{R}^m$. According to Baire's Lemma (see [20, p.15]), there exists a sequence $\{(q^k, b^k)\} \subset \mathbb{R}^n \times \mathbb{R}^m$ converging to $(\bar{q}, \bar{b})$ such that $(-q^k, -b^k) \notin \mathcal{P}$ for all k.

From the assumption that $S(\bar{Q}, ., \bar{D}, \bar{r}, \bar{A}, .)$ is lower semicontinuous at $(\bar{q}, \bar{b})$, $S(\bar{w})$ is nonempty. Fix any $\bar{x} \in S(\bar{w})$. Then, there exists a sequence $\{x^k\} \subset \mathbb{R}^n$ converging to $\bar{x}$ such that $x^k \in S(\bar{Q}, q^k, \bar{D}, \bar{r}, \bar{A}, b^k)$ for all k. For each k, there exists $(\lambda^k, \mu^k) \in \mathbb{R} \times \mathbb{R}^m$ such that:

$$(\bar{Q} + \lambda^k \bar{D})x^k + (\mu^k)^T \bar{A} + q^k = 0, \tag{4.2}$$

$$\lambda^k \geq 0,\ \mu^k \geq 0,\ (x^k)^T \bar{D} x^k - \bar{r}^2 \leq 0,\ \bar{A}x^k + b^k \leq 0, \tag{4.3}$$

$$\lambda^k((x^k)^T \bar{D} x^k - \bar{r}^2) = 0,\ \mu_i^k(\bar{A}_i x^k + b_i^k) = 0,\ \ i = 1, \ldots, m. \tag{4.4}$$

Let $S_k = \{i \in \{1, \ldots, m\} : \mu_i^k > 0\}$. Then, there exists a subset $J \subset \{1, \ldots, m\}$ such that $S_k = J$ for infinitely many k. Without loss of generality we may assume that $S_k = J$ for every k. Hence (4.2)–(4.4) reduces to

$$(\bar{Q} + \lambda^k \bar{D})x^k + \bar{A}_J^T \mu_J^k + q^k = 0,$$
$$\bar{A}_J x^k + b_J^k = 0.$$

This can be rewritten as follows

$$\begin{pmatrix} -q^k \\ -b_J^k \end{pmatrix} = \mathcal{A}_{0J}(\lambda^k) \begin{pmatrix} x^k \\ \mu_J^k \end{pmatrix}. \tag{4.5}$$

Consider the following three cases:

Case 1: $\lambda^k = 0$ for infinitely many k. There is no loss of generality in assuming $\lambda^k = 0$ for every k. From (4.5) it follows

$$\begin{pmatrix} -q^k \\ -b_J^k \end{pmatrix} = \mathcal{A}_J \begin{pmatrix} x^k \\ \mu_J^k \end{pmatrix}. \tag{4.6}$$

This gives $(-q^k, -b^k) \in \mathcal{P}_J$. If $\det \mathcal{A}_J = 0$ then $(-q^k, -b^k) \in \mathcal{P}$, contrary to the fact that $(-q^k, -b^k) \notin \mathcal{P}$. Hence $\det \mathcal{A}_J \neq 0$ and

$$\begin{pmatrix} x^k \\ \mu_J^k \end{pmatrix} = (\mathcal{A}_J)^{-1} \begin{pmatrix} -q^k \\ -b_J^k \end{pmatrix}$$

follows from (4.6). It implies that μ_J^k converges to some $\bar{\mu}_J \in \mathbb{R}$. Hence

$$\begin{pmatrix} \bar{x} \\ \bar{\mu}_J \end{pmatrix} = (\mathcal{A}_J)^{-1} \begin{pmatrix} -\bar{q} \\ -\bar{b}_J \end{pmatrix}.$$

Therefore $\bar{x}$ is defined uniquely by J.

Case 2: $\lambda^k \in T$ for infinitely many k. Since T is finite, $\lambda^k = \bar{\lambda}$ for infinitely many k, for some $\bar{\lambda} \in T$. There is no loss of generality in assuming that $\lambda^k = \bar{\lambda}$ for every k. Then, (4.5) leads to

$$\begin{pmatrix} -q^k \\ -b^k \end{pmatrix} = \mathcal{A}_{0J}(\bar{\lambda}) \begin{pmatrix} x^k \\ \mu^k \end{pmatrix}.$$

Combining this with $\bar{\lambda} \in T$ gives $(-q^k, -b^k) \in \mathcal{P}$, contrary to the fact that $(-q^k, -b^k) \notin \mathcal{P}$. Thus this case does not occur.

Case 3: $\lambda^k \notin T \cup \{0\}$ for infinitely many k. There is no loss of generality in assuming that $\lambda^k \notin T \cup \{0\}$ for every k. Since λ^k is not an eigenvalue of $C^{-1}\bar{Q}C$, we obtain that

$$\det(\bar{Q} + \lambda^k \bar{D}) = \det(C^{-1}\bar{Q}C + \lambda^k I) \neq 0$$

and

$$\det \mathcal{A}_{0J}(\lambda^k) = \det(\bar{Q} + \lambda^k \bar{D}) \det(-\bar{A}_J^T(\bar{Q} + \lambda^k \bar{D})^{-1}\bar{A}_J).$$

By the assumption that $\bar{A}$ has full rank, so is $\bar{A}_J$. Then,

$$\mathrm{rank}(\bar{A}_J^T(\bar{Q} + \lambda^k \bar{D})^{-1}\bar{A}_J) = |J|,$$

that is, $\det(\bar{A}_J^T(\bar{Q} + \lambda^k \bar{D})^{-1}\bar{A}_J) \neq 0$. This leads to $\det \mathcal{A}_{0J}(\lambda^k) \neq 0$ for every k. From (4.5) we get

$$\begin{pmatrix} x^k \\ \mu_J^k \end{pmatrix} = (\mathcal{A}_{0J}(\lambda^k))^{-1} \begin{pmatrix} -q^k \\ -b_J^k \end{pmatrix}. \tag{4.7}$$

From the assumption that $(ET_m(\bar{w}))$ satisfies (SCQ), $(ET(w^k))$ also satisfies (SCQ) for every k large enough. According to Lemma 2.3, for each k large enough, $\{(\lambda^k, \mu^k)\}$ is bounded. Hence $\{(\lambda^k, \mu_J^k)\}$ is bounded. Without loss of generality, assume that $(\lambda^k, \mu_J^k) \to (\hat{\lambda}, \hat{\mu}_J)$ for some $(\hat{\lambda}, \hat{\mu}_J) \in \mathbb{R} \times \mathbb{R}^{|J|}$. Then, the sequence on the right hand side of (4.7) is convergent. Passing both sides of the equality (4.7) to the limits as $k \to \infty$, we deduce that $\bar{x}$ is defined uniquely by J.

In all above four cases, $\bar{x}$ is defined uniquely by J, for some $J \subset \{1, \ldots, m\}$. Therefore the number of elements of $S(\bar{w})$ can not be greater than 2^m. The proof is complete. $\square$

From Theorem 4.1 it follows immediately the following corollary.

Corollary 4.1. *Consider the problem* $(ET_m(w))$ *and* $\bar{w} \in \mathbb{W}$*. If* $\bar{A}$ *has full rank and* $S(\cdot)$ *is lower semicontinuous at* $\bar{w}$ *then* $(ET_m(\bar{w}))$ *satisfies (SCQ) and* $S(\bar{w})$ *is a nonempty set which contains at most* 2^m *points.*

Denote

$$\partial\mathcal{F}(\bar{w}) := \Big\{x \in \mathcal{F}(\bar{w}) : (x^T\bar{D}x - \bar{r}^2)\prod_{i=1}^{m}(\bar{A}_i x - \bar{b}_i) = 0\Big\}.$$

The following theorem shows some sufficient conditions for the lower semicontinuity of $S(\cdot)$.

Theorem 4.2. *Consider* $(ET_m(w))$ *and* $\bar{w} = (\bar{Q}, \bar{q}, \bar{D}, \bar{r}, \bar{A}, \bar{b}) \in \mathbb{W}$*. If* $S(\bar{w}) \neq \emptyset$ *and at least one of the following conditions is satisfied:*

(i) $\bar{Q}+\lambda\bar{D}$ *is positive definite for every KKT pair* (x, λ, μ) *and* $(ET_m(\bar{w}))$ *satisfies (SCQ);*

(ii) $S(\bar{w})$ *is a singleton and* $(ET_m(\bar{w}))$ *satisfies (SCQ);*

(iii) $S(\bar{w})$ *is a singleton and* φ *is continuous at* $\bar{w}$*;*

(iv) $G(\cdot)$ *is lower semicontinuous at* $\bar{w}$*;*

(v) $S(\bar{w})$ *is finite and* $S(\bar{w}) \cap \partial\mathcal{F}(\bar{w}) = \emptyset$*;*

(vi) $\bar{Q}$ *is nonsingular and* $S(\bar{w}) \cap \partial\mathcal{F}(\bar{w}) = \emptyset$*,*

then $S(\cdot)$ *is lower semicontinuous at* $\bar{w}$*.*

Proof. In oder to prove that $S(\cdot)$ is lower semicontinuous at $\bar{w}$, we have to show that for any $z \in S(\bar{w})$ and for any open neighborhood U_z of z, there exists $\delta > 0$ such that

$$S(\tilde{w}) \cap U_z \neq \emptyset \tag{4.8}$$

for every $\tilde{w} \in \mathbb{W}$ satisfying $\|\tilde{w} - \bar{w}\| < \delta$.

We now fix any $x \in S(\bar{w})$ with the corresponding Lagrange multiplier (λ, μ). Let U_x be an open neighborhood of x.

Firstly, suppose that (i) holds. Let

$$L(y, \bar{w}, \lambda, \mu) := f(y, \bar{w}) + \lambda h_0(y, \bar{w}) + \sum_{i=1}^{m} \mu_i h_i(y, \bar{w}).$$

From system (2.4)–(2.6) it follows that $\nabla L_y(x, \bar{w}, \lambda, \mu) = 0$. For every $\tilde{x} \in \mathcal{F}(\bar{w})$ and $\tilde{x} \neq x$, we have

$$\begin{aligned} f(\tilde{x}, \bar{w}) - f(x, \bar{w}) &\geq L(\tilde{x}, \bar{w}, \lambda, \mu) - L(x, \bar{w}, \lambda, \mu) \\ &= \frac{1}{2}(\tilde{x} - x)^T(\bar{Q} + \lambda\bar{D})(\tilde{x} - x) + \nabla L_y(x, \bar{w}, \lambda, \mu)^T(\tilde{x} - x) \\ &> 0 \end{aligned}$$

by the assumption that $\bar{Q} + \lambda D$ is positive definite. Hence x is the unique solution of the problem $(ET_m(w))$ with $w = \bar{w}$.

According to Theorem 2.1, $G(\cdot)$ is upper semicontinuous at $\bar{w}$. Hence there exists $\epsilon^3 > 0$ such that $G(\tilde{w}) \cap U_x \neq \emptyset$ for every $\tilde{w} \in \mathbb{W}$ satisfying $\|\tilde{w} - \bar{w}\| < \epsilon^3$. The latter leads to (4.8).

We now suppose that (ii) holds, i.e., $(ET_m(\bar{w}))$ satisfies (SCQ) and $S(\bar{w}) = \{x\}$. According to Lemma 2.1, there exists $\delta > 0$ such that $\mathcal{F}(\tilde{w}) \neq \emptyset$ for every $\tilde{w}$ satisfying $\|\tilde{w} - \bar{w}\| < \delta$. Since $\mathcal{F}(\tilde{w})$ is nonempty and compact, $S(\tilde{w}) \neq \emptyset$ for every $\tilde{w}$ satisfying $\|\tilde{w} - \bar{w}\| < \delta$. By Theorem 2.4, we have $S(\cdot)$ is upper semicontinuous at $\bar{w}$. Hence $S(\tilde{w}) \subset U_x$ for every $\tilde{w}$ satisfying $\|\tilde{w} - \bar{w}\| < \delta$. It follows that $S(\tilde{w}) \cap U_x \neq \emptyset$ for every $\tilde{w}$ satisfying $\|\tilde{w} - \bar{w}\| < \delta$. Thus $S(\cdot)$ is lower semicontinuous at $\bar{w}$.

By (ii) and Theorem 4.3, we obtain (iii).

The assertion (iv) follows from (ii) and Theorem 2.2.

We next consider the case where (v) holds, i.e., $x^T \bar{D} x < \bar{r}^2$, $\bar{A}x + \bar{b} < 0$ and $S(\bar{w})$ is finite. It follows that $(\lambda, \mu) = (0, 0) \in \mathbb{R} \times \mathbb{R}^m$ and x is a solution of the following linear system

$$\bar{Q}y = -\bar{q}. \tag{4.9}$$

Since $x^T\bar{D}x < \bar{r}^2$ and $\bar{A}x + \bar{b} < 0$, there exist $\epsilon^1 > 0$ and an open neighborhood $V_x \subset U_x$ such that $V_x \subset \mathcal{F}(\tilde{w})$ for every $\tilde{w}$ satisfying $\|\tilde{w} - \bar{w}\| < \epsilon^1$.

Since $S(\bar{w})$ is finite, from (4.9) it follows that $\bar{Q}$ is nonsingular and x is a unique solution of (4.9). This gives $x = -\bar{Q}^{-1}\bar{q}$. Then, there exists $\epsilon^2 > 0$ such that $\tilde{x} = -\tilde{Q}^{-1}\tilde{q} \in V_x$ for every $(\tilde{Q}, \tilde{q})$ satisfying $\|(\tilde{Q}, \tilde{q}) - (\bar{Q}, \bar{q})\| \leq \|\tilde{w} - \bar{w}\| < \epsilon^2$.

Let $\epsilon = min\{\epsilon^1, \epsilon^2\}$ and let $\tilde{w} \in \mathbb{W}$ satisfying $\|\tilde{w} - \bar{w}\| < \epsilon$. Then, $\tilde{x} \in V_x \subset \mathcal{F}(\tilde{w})$ and $(\lambda, \mu) = (0, 0) \in \mathbb{R} \times \mathbb{R}^m$ is the unique Lagrange multiplier corresponding to $\tilde{x}$. We have

$$\tilde{Q}\tilde{x} + \tilde{q} = 0,\ \tilde{x}^T\tilde{D}\tilde{x} - \tilde{r}^2 < 0,\ \tilde{A}\tilde{x} + \tilde{b} < 0.$$

Hence (4.8) is satisfied for every $\tilde{w} \in \mathbb{W}$ satisfying $\|\tilde{w} - \bar{w}\| < \epsilon$.

Finally, we consider the case where (vi) holds, i.e., $x^T\bar{D}x < \bar{r}^2$, $\bar{A}x + \bar{b} < 0$ and $\bar{Q}$ is nonsingular. Repeating the previous arguments and using the assumption that $\bar{Q}$ is nonsingular lead to (4.8). This completes the proof of the theorem. □

By Theorem 4.2, we obtain the following corollary.

Corollary 4.2. *Consider $(ET_m(w))$ and $\bar{w} \in \mathbb{W}$. If $(ET_m(\bar{w}))$ satisfies (SCQ) and $\bar{Q}$ is positive definite, then $S(\cdot)$ is lower semicontinuous at $\bar{p}$.*

Proof. Since $\bar{Q}$ is positive definite, $G(\bar{p})$ is a singleton and $G(\bar{p}) = S(\bar{p})$. By the part (i) of Theorem 4.2, $S(\cdot)$ is lower semicontinuous at $\bar{p}$. □

We conclude this section by a simple example showing that $S(\cdot)$ is not lower semicontinuous at $\bar{w}$ if it is infinite.

Example 4.1. *Consider the problem $(ET_m(w))$ with $n = 2, m = 1$,*

$$\bar{Q} = -I, \quad \bar{q} = 0, \quad \bar{D} = I, \quad \bar{r} = 1, \quad \bar{A}_1 = (-1, -1), \quad \bar{b}_1 = 1.$$

This problem has the following form

$$\min\left\{f(x,\bar{w}) = -\frac{1}{2}(x_1^2+x_2^2) : x \in \mathcal{F}(\bar{w})\right\},$$

where

$$\mathcal{F}(\bar{w}) = \{(x_1,x_2)\in\mathbb{R}^2 : x_1^2+x_2^2 \le 1; -x_1-x_2+1\le 0\}.$$

We obtain that $(x,1,0)$ *is a KKT pair for every* $x \in \{(x_1,x_2)\in\mathbb{R}^2 : x_1^2+x_2^2 = 1; x_1\ge 0; x_2\ge 0\}$*. Hence* $S(\bar{w})$ *is infinite. By Theorem 4.1,* $S(\cdot)$ *is not lower semicontinuous at* $\bar{w}$*.*

4.2.2. Continuity of the optimal value function

The main result in this subsection is presented in the following theorem.

Theorem 4.3. *Consider* $(ET_m(w))$ *and* $\bar{w} = (\bar{Q},\bar{q},\bar{D},\bar{r},\bar{A},\bar{b})\in\mathbb{W}$*. The following assertions hold:*

(i) φ *is lower semicontinuous at* $\bar{w}$*;*

(ii) φ *is upper semicontinuous at* $\bar{w}$ *if* $(ET_m(\bar{w}))$ *satisfies (SCQ);*

(iii) *If* $\mathcal{F}(\bar{w})$ *is nonempty and if* φ *is continuous at* $\bar{w}$*, then* $(ET_m(\bar{w}))$ *satisfies (SCQ);*

(iv) *If* $\mathcal{F}(\bar{w})$ *is empty, then* φ *is continuous at* $\bar{w}$*.*

Proof. (i) Let any sequence $\{w^k\}\subset\mathbb{W}$ such that $w^k\to\bar{w}$. We have to show that $\liminf_{k\to\infty}\ \varphi(w^k)\ge\varphi(\bar{w})$.

Suppose, on the contrary, that

$$\liminf_{k\to\infty}\ \varphi(w^k) < \varphi(\bar{w}).$$

Without loss of generality, we may assume that

$$\liminf_{k\to\infty}\ \varphi(w^k) = \lim_{k\to\infty}\ \varphi(w^k).$$

Then, there exist a real number $\delta > 0$ and an index k_0 such that

$$\varphi(w^k) \leq \delta < \varphi(\bar{w})$$

for every $k \geq k_0$. Since $\varphi(w^k) < +\infty$, we obtain that $\mathcal{F}(w^k) \neq \emptyset$ and $G(w^k) \neq \emptyset$. Hence there exists a bounded sequence $\{x^k\}$ such that $x^k \in \mathcal{F}(w^k)$. We may assume that this sequence itself converges to a vector $\hat{x} \in \mathcal{F}(\bar{w})$. Since $x^k \in \mathcal{F}(w^k)$, we have

$$\varphi(w^k) = f(x^k, w^k) \leq \delta.$$

Letting $k \to \infty$, we get $f(\hat{x}, \bar{w}) \leq \delta$. Hence

$$\delta < \varphi(\bar{w}) \leq f(\hat{x}, \bar{w}) \leq \delta.$$

This is impossible. Therefore $\liminf_{k\to\infty} \varphi(w^k) \geq \varphi(\bar{w})$.

(ii) Since $(ET_m(\bar{w}))$ satisfies (SCQ), we get $\mathcal{F}(\bar{w}) \neq \emptyset$. By Theorem 3.1, we obtain that φ is upper semicontinuous at $\bar{w}$.

(iii) This assertion follows from Theorem 3.1.

(iv) Suppose that the set $\mathcal{F}(\bar{w})$ is empty. We shall show that, for every sequence $\{w^k = (Q^k, q^k, D^k, r^k, A^k, b^k)\} \subset \mathbb{W}$ converging to $\bar{w}$,

$$\liminf_{k\to\infty} \varphi(w^k) = +\infty.$$

Suppose that $\liminf_{k\to\infty} \varphi(w^k) < +\infty$. Then, there exist a number $\beta > 0$ and a subsequence $\{w^s\}$ of $\{w^k\}$ such that

$$\varphi(w^s) \leq \beta \ \ \forall s \in \mathbb{N}.$$

This leads to $G(w^s) \neq \emptyset$. For each s, there exists $x^s \in G(w^s)$, that is,

$$\varphi(w^s) \leq \beta;\ (x^s)^T D^s x^s \leq (r^s)^2;\ A^s x^s + b^s \leq 0. \tag{4.10}$$

Since $\{x^s\}$ is bounded, without loss of generality, we may assume that $\{x^s\}$ converges to $\hat{x} \in \mathbb{R}^n$. Passing the second and the third inequalities in (4.10) to the limits as $s \to +\infty$, we obtain

$$\hat{x}^T \bar{D} \hat{x} \leq \bar{r}^2;\ \bar{A}\hat{x} + \bar{b} \leq 0.$$

This follows $\mathcal{F}(\bar{w}) \neq \emptyset$, contrary to the assumption $\mathcal{F}(\bar{w}) = \emptyset$. The theorem is proved. □

4.3. ETRS with a linear inequality constraint

In this section, we concern the problem $(ET_1(w))$ with $D = I$. By the special structure and by using the tools from variational analysis, we obtain some interesting results on stability of this problem.

4.3.1. Lower semicontinuity of the stationary solution map

Our main purpose in this subsection is to establish necessary and sufficient conditions for the lower semicontinuities of the stationary solution map to parametric $(ET_1(w))$.

A necessary and sufficient condition for the lower semicontinuity of the stationary solution map $(\tilde{q}, \tilde{b}) \mapsto S(Q, \tilde{q}, r, a, \tilde{b})$ is proposed below.

Theorem 4.4. *Consider* $(ET_1(w))$ *and* $\bar{w} = (\bar{Q}, \bar{q}, \bar{r}, \bar{a}, \bar{b}) \in \mathbb{W}$. *The multifunction* $S(\bar{Q}, ., \bar{r}, \bar{a}, .)$ *is lower semicontinuous at* $(\bar{q}, \bar{b})$ *if and only if* $(ET_1(\bar{w}))$ *satisfies* (SCQ) *and* $S(\bar{w})$ *is a singleton.*

Proof. Necessity: Repeating the argument in the proof of Theorem 4.1, we obtain that $(ET_1(\bar{w}))$ satisfies (SCQ).

For each $t \in \mathbb{R}$, we set:

$$
\begin{aligned}
M_{\emptyset}(t) &= \bar{Q};\\
M_1(t) &= \bar{Q} + tI;\\
M_2(t) &= \begin{pmatrix} \bar{Q} & \bar{a} \\ \bar{a}^T & 0 \end{pmatrix};\\
M_{1,2}(t) &= \begin{pmatrix} \bar{Q} + tI & \bar{a} \\ \bar{a}^T & 0 \end{pmatrix}.
\end{aligned}
$$

For each $J \in \{\{1\}, \{1, 2\}\}$, we denote by T_J the solutions set of the equation $detM_J(t) = 0$ in variable t. Since T_1 is the set of all eigenvalues of $\bar{Q}$, T_1 is finite. For $t \notin T_1$, we have

$$\det M_{1,2}(t) = \det(\bar{Q} + tI). \det(-\bar{a}^T(\bar{Q} + tI)^{-1}\bar{a}).$$

If $\bar{Q} = D = \text{diag}\{d_1, ..., d_n\},\ d_1 \le ... \le d_n$, then

$$\det M_{1,2}(t) = \prod_{i=1}^{n}(d_i + t)\sum_{i=1}^{n}\frac{\bar{a}_i^2}{d_i + t}.$$

Otherwise, there exists an orthogonal matrix P such that

$$P^{-1}\bar{Q}P = D = \text{diag}\{d_1, ..., d_n\},\ d_1 \le ... \le d_n.$$

Then,

$$\det M_{1,2}(t) = \det(D + tI).\det(-b^T(D + tI)^{-1}b),$$

with $b = P^T\bar{a}$ and

$$\det M_{1,2}(t) = \prod_{i=1}^{n}(d_i + t)\sum_{i=1}^{n}\frac{b_i^2}{d_i + t}.$$

We have $b \neq 0$ if $\bar{a} \neq 0$ (since $\det P \neq 0$). Hence $\det M_{1,2}(t)$ is a polynomial of degree $n - 1$ in variable t as $\bar{a} \neq 0$. Thus $T_{1,2}$ is a finite set. Let $T = T_1 \cup T_{1,2}$. Then, T is finite.

For each $J \in \{\emptyset, \{1\}\}$, let

$$\Gamma_J(t) = \Big\{(u, v) \in \mathbb{R}^n \times \mathbb{R} : u = M_J(t)x \text{ for some } x \in \mathbb{R}^n\Big\}.$$

For each $J \in \{\{2\}, \{1, 2\}\}$, let

$$\Gamma_J(t) := \Big\{(u, v) \in \mathbb{R}^n \times \mathbb{R} : \begin{pmatrix} u \\ v \end{pmatrix} = M_J(t)\begin{pmatrix} x \\ \mu \end{pmatrix} \text{ for some } (x, \mu) \in \mathbb{R}^n \times \mathbb{R}^m\Big\}.$$

Let

$$\Gamma = \bigcup\big\{\Gamma_J(t) : J \subset \{1, 2\}, t \in T, \det M_J(t) = 0\big\}.$$

For any $J \subset \{1, 2\}$, if $\det M_J(t) = 0$, then $\Gamma_J(t)$ is a proper linear subspace of $\mathbb{R}^n \times \mathbb{R}$. According to Baire's lemma (see [20, p.15]), Γ is nowhere dense in $\mathbb{R}^n \times \mathbb{R}$. Hence there exists a sequence $\{(q^k, b^k)\}$ converging to $(\bar{q}, \bar{b})$ such that $(-q^k, -b^k) \notin \Gamma$ for all k. Since $S(\bar{Q}, ., \bar{r}, \bar{a}, .)$ is lower semicontinuous at $(\bar{q}, \bar{b})$, $S(\bar{w})$ is nonempty. Fix any $\bar{x} \in S(\bar{p})$. Since $S(\bar{Q}, ., \bar{r}, \bar{a}, .)$ is lower semicontinuous at $(\bar{q}, \bar{b})$, without loss of generality, we can assume that there exists a sequence $\{x^k\} \subset \mathbb{R}^n$ converging to $\bar{x}$ such that $x^k \in S(\bar{Q}, q^k, \bar{r}, \bar{a}, b^k)$ for all k. Then, for each k, there exists $(\lambda^k, \mu^k) \in \mathbb{R}^2$ such that

$$(\bar{Q} + \lambda^k I)x^k + \mu^k \bar{a} + q^k = 0, \tag{4.11}$$

$$\lambda^k \geq 0,\ \mu^k \geq 0,\ \|x^k\| - \bar{r} \leq 0,\ \bar{a}^T x^k + b^k \leq 0, \tag{4.12}$$

$$\lambda^k \big(\|x^k\| - \bar{r}\big) = 0,\ \mu^k(\bar{a}^T x^k + b^k) = 0. \tag{4.13}$$

For each k, let

$$J_k = \begin{cases} \emptyset & \text{if } \mu_k \leq 0, \\ \{2\} & \text{if } \mu_k > 0 \text{ and } \lambda_k < 0, \\ \{1; 2\} & \text{if } \mu_k > 0 \text{ and } \lambda_k \geq 0. \end{cases}$$

Then, there exists a set $K \in \big\{\emptyset, \{2\}, \{1, 2\}\big\}$ such that $J_k = K$ for infinitely many k. Without loss of generality we can assume that $J_k = K$ for all k. We distinguish the following three cases:

Case 1: $K = \emptyset$. Then, the system (4.31)–(4.4) reduces to

$$\bar{Q}x^k + q^k = 0, \tag{4.14}$$

that is,

$$-q^k = M_\emptyset(t)x^k.$$

Hence $(-q^k, -b^k) \in \Gamma_\emptyset(t)$. If $\det \bar{Q} = \det M_\emptyset(t) = 0$ then $(-q^k, -b^k) \in \Gamma$, contrary to the fact that $(-q^k, -b^k) \notin \Gamma$. Thus $\det \bar{Q} \neq 0$. From (4.14) it follows

$$x^k = -\bar{Q}^{-1}q^k.$$

Letting $k \to \infty$, we have $\bar{x} = -\bar{Q}^{-1}\bar{q}$. Since $\bar{x}$ is chosen arbitrarily, $S(\bar{w})$ is a singleton.

Case 2: $K = \{2\}$. Then, the system (4.11)–(4.13) reduces to

$$\bar{Q}x^k + \mu^k \bar{a} + q^k = 0, \tag{4.15}$$

$$\bar{a}^T x^k + b^k = 0, \tag{4.16}$$

that is,

$$\begin{pmatrix} -q^k \\ -b^k \end{pmatrix} = M_2(t) \begin{pmatrix} x^k \\ \mu^k \end{pmatrix}. \tag{4.17}$$

This implies $(-q^k, -b^k) \in \Gamma_2(t)$. If $\det M_2(t) = 0$ then $(-q^k, -b^k) \in \Gamma$, contrary to the fact that $(-q^k, -b^k) \notin \Gamma$. Hence $\det M_2(t) \neq 0$. From (4.17) it follows that

$$\begin{pmatrix} x^k \\ \mu^k \end{pmatrix} = (M_2(t))^{-1} \begin{pmatrix} -q^k \\ -b^k \end{pmatrix}.$$

Hence μ^k converges to some $\mu^0 \in \mathbb{R}$. Since x^k converges to $\bar{x}$, we have

$$\begin{pmatrix} \bar{x} \\ \mu^0 \end{pmatrix} = (M_2(t))^{-1} \begin{pmatrix} -\bar{q} \\ -\bar{b} \end{pmatrix}.$$

From this it follows that $\bar{x}$ is defined uniquely and $S(\bar{w})$ is a singleton.

Case 3: $K = \{1, 2\}$. Then, the system (4.11)-(4.13) reduces to

$$(\bar{Q} + \lambda^k I)x^k + \mu^k \bar{a} + q^k = 0, \tag{4.18}$$

$$\bar{a}^T x^k + b^k = 0, \tag{4.19}$$

that is,

$$\begin{pmatrix} -q^k \\ -b^k \end{pmatrix} = M_{1,2}(\lambda^k) \begin{pmatrix} x^k \\ \mu^k \end{pmatrix}. \tag{4.20}$$

Consider the following two subcases:

Subcase 3.1: $\lambda^k \notin T$ for infinitely many k. Without loss of generality we can assume that $\lambda^k \notin T$ for all k. Then, $\det M_{1,2}(\lambda^k) \neq 0$ for every k. By (4.20), we obtain

$$\begin{pmatrix} x^k \\ \mu^k \end{pmatrix} = (M_{1,2}(\lambda^k))^{-1} \begin{pmatrix} -q^k \\ -b^k \end{pmatrix}. \tag{4.21}$$

By Lemma 2.4, we obtain that $\{(\lambda^k, \mu^k)\}$ is bounded. Without loss of generality, one may assume that $(\lambda^k, \mu^k) \to (\hat{\lambda}, \hat{\mu})$ for some $(\hat{\lambda}, \hat{\mu}) \in \mathbb{R}^2$. Thus the sequence on the right hand side of (4.21) is convergent.

Taking limitation both sides of equality (4.21) as $k \to \infty$, we obtain that $\bar{x}$ is defined uniquely. Since $\bar{x}$ is chosen arbitrarily, $S(\bar{w})$ is a singleton.

Subcase 3.2: $\lambda^k \in T$ for infinitely many k. Without loss of generality, we can assume that $\lambda^k \in T$ for all k. Since T is finite, we can assume that $\lambda^k = \bar{\lambda}$ for some $\bar{\lambda} \in T$, for every k. From (4.20),

$$\begin{pmatrix} -q^k \\ -b^k \end{pmatrix} = M_{1,2}(\bar{\lambda}) \begin{pmatrix} x^k \\ \mu^k \end{pmatrix}. \tag{4.22}$$

Since $\det M_{1,2}(\bar{\lambda}) = 0$, (4.22) implies that $(-q^k, -b^k) \in \Gamma$. This contradicts the fact that $(-q^k, -b^k) \notin \Gamma$. Hence this subcase does not occur.

Sufficiency: Suppose that $(ET_1(\bar{w}))$ satisfies (SCQ) and $S(\bar{w})$ is a singleton. Let U be an open set containing $\bar{x} \in S(\bar{w})$. Since $(ET_1(\bar{w}))$ satisfies (SCQ), there exists $\delta_1 > 0$ such that $\mathcal{F}(\bar{Q}, \bar{q}, \bar{r}, \bar{a}, \tilde{b}) \neq \emptyset$ for every $\tilde{b}$ satisfying $\|\tilde{b} - \bar{b}\| < \delta_1$ (see Lemma 2.1). Since $\mathcal{F}(\bar{Q}, \bar{q}, \bar{r}, \bar{a}, \tilde{b})$ is nonempty and compact, $S(\bar{Q}, \tilde{q}, \bar{r}, \bar{a}, \tilde{b}) \neq \emptyset$ for every pair $(\tilde{q}, \tilde{b})$ satisfying $\|(\tilde{q}, \tilde{b}) - (\bar{q}, \bar{b})\| < \delta_1$.

By Theorem 2.4 and by the assumption that $(ET_1(\bar{w}))$ satisfies (SCQ), we have $S(\bar{Q}, ., \bar{r}, \bar{a}, .)$ is upper semicontinuous at $(\bar{q}, \bar{b})$, that is, there exists $\delta_2 > 0$ such that $S(\bar{Q}, \tilde{q}, \bar{r}, \bar{a}, \tilde{b}) \subset U$ for every pair $(\tilde{q}, \tilde{b})$ satisfying $\|(\tilde{q}, \tilde{b}) - (\bar{q}, \bar{b})\| < \delta_2$. Let $\delta = \min\{\delta_1; \delta_2\}$. Then, we obtain $S(\bar{Q}, \tilde{q}, \bar{r}, \bar{a}, \tilde{b}) \cap U \neq \emptyset$ for every pair $(\tilde{q}, \tilde{b})$ satisfying $\|(\tilde{q}, \tilde{b}) - (\bar{q}, \bar{b})\| < \delta$. Thus $S(\bar{Q}, ., \bar{r}, \bar{a}, .)$ is lower semicontinuous at $(\bar{q}, \bar{b})$. The proof is then complete. $\square$

Theorem 4.4 leads to the following result which characterizes the lower semicontinuity of the stationary solution map under total perturbations.

Theorem 4.5. *Consider $(ET_1(w))$ and $\bar{w} = (\bar{Q}, \bar{q}, \bar{D}, \bar{r}, \bar{A}, \bar{b}) \in \mathbb{W}$. The multifunction $\tilde{w} \mapsto S(\tilde{w})$ is lower semicontinuous at $\bar{w}$ if and only if $(ET_1(\bar{w}))$ satisfies (SCQ) and $S(\bar{w})$ is a singleton.*

Proof. Suppose that $S(\cdot)$ is lower semicontinuous at $\bar{w}$. Then, the multifunction $(\tilde{q}, \tilde{b}) \mapsto S(\bar{Q}, \tilde{q}, \bar{r}, \bar{a}, \tilde{b})$ is lower semicontinuous at $(\bar{q}, \bar{b})$. By Theorem 4.4, $(ET_1(\bar{w}))$ satisfies (SCQ) and $S(\bar{w})$ is a singleton.

Conversely, suppose that $(ET_1(\bar{w}))$ satisfies (SCQ) and $S(\bar{w})$ is a singleton. By repeating the argument in the proof of Theorem 4.4 for w be perturbed, we have the desired conclusion. □

The following corollary shows the closed relation between the stationary solution set map $S(\cdot)$ and the global optimal solution map $G(\cdot)$.

Corollary 4.3. *Consider the problem $(ET_1(w))$ and $\bar{w} \in \mathbb{W}$. Assume that $(ET_1(\bar{w}))$ satisfies (SCQ). If $S(\cdot)$ is continuous at $\bar{w}$ then $G(\cdot)$ is continuous at $\bar{w}$.*

Proof. The desired conclusion follows immediately from Theorem 2.2 and Corollary 4.5. □

We now give some examples to illustrate the above results.

Example 4.2. *Consider the problem $(ET_1(\bar{w}))$ with $n = 2$ and*

$$\bar{Q} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \bar{q} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

$$\bar{r} = 1, \bar{a} = \begin{pmatrix} -\sqrt{3} + 2\sqrt{2} + 1 \\ -\sqrt{3} - 1 \end{pmatrix}, \bar{b} = 2 + \sqrt{2}.$$

We can rewrite this problem as follows

$$\min \left\{ f(x, \bar{w}) = \frac{1}{2}(x_1^2 - x_2^2) : \ x \in \mathcal{F}(w) \right\},$$

where

$$\begin{aligned} \mathcal{F}(\bar{w}) = \{(x_1, x_2) \in \mathbb{R}^2 : \ & x_1^2 + x_2^2 \leq 1; \\ & (-\sqrt{3} + 2\sqrt{2} + 1)x_1 + (-\sqrt{3} - 1)x_2 + 2 + \sqrt{2} \leq 0\}. \end{aligned}$$

We can check that (SCQ) holds at $\bar{w}$. Solving the KKT condition, we get

$$S(\bar{p}) = \Big\{ \Big(-\frac{1}{2}, \frac{\sqrt{3}}{2} \Big) \Big\}.$$

From Theorems 4.1 and 4.3 it follows that both $S(\cdot)$ and $G(\cdot)$ are continuous at $\bar{w}$.

Example 4.3. *Consider the problem $(ET_1(\bar{w}))$ with $n = 2$ and*

$$\bar{Q} = \begin{pmatrix} 1 & 0 \\ 0 & -3 \end{pmatrix}, \quad \bar{q} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \bar{r} = 1, \quad \bar{a} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}, \ \bar{b} = -2.$$

This problem can be rewritten as follows

$$\min \Big\{ f(x, \bar{w}) = \frac{1}{2}(x_1^2 - 3x_2^2) : \ x \in \mathcal{F}(\bar{w}) \Big\},$$

where

$$\mathcal{F}(\bar{w}) = \{(x_1, x_2) \in \mathbb{R}^2 : \ x_1^2 + x_2^2 \leq 1; \ -2x_1 + x_2 - 2 \leq 0\}.$$

It is easy to verify that (SCQ) holds at $\bar{w}$ and

$$f(x, \bar{w}) \geq -\frac{3}{2}$$

for every $x \in \mathcal{F}(\bar{w})$. On the other hand,

$$f((0,1), \bar{w}) = f((0,-1), \bar{w}) = -\frac{3}{2}.$$

Hence the number of elements of the set $G(\bar{w})$ is greater than 1. By Theorems 4.1 and 4.3, both $G(\cdot)$ and $S(\cdot)$ are not lower semicontinuous at $\bar{w}$.

4.3.2. Coderivatives of the normal cone mapping

Let us recall some facts from [73]. The Fréchet normal cone to a set $\Omega \subset \mathbb{R}^n$ at $\bar{x} \in \Omega$ is given by

$$\widehat{N}(\bar{x}; \Omega) := \Big\{ x^* \in \mathbb{R}^n : \limsup_{x \xrightarrow{\Omega} \bar{x}} \frac{\langle x^*, x - \bar{x} \rangle}{\|x - \bar{x}\|} \leq 0 \Big\},$$

where $x \xrightarrow{\Omega} \bar{x}$ means $x \to \bar{x}$ with $x \in \Omega$. By convention, $\widehat{N}(\bar{x};\Omega) = \emptyset$ when $\bar{x} \notin \Omega$. For a multifunction $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$, the *sequential Painlevé-Kuratowski upper limit* with respect to the norm topology of $\mathbb{R}^n$ is defined by

$$\operatorname*{Lim\,sup}_{x \xrightarrow{\Omega} \bar{x}} F(x) := \Big\{x^* \in \mathbb{R}^n : \exists x_k \to \bar{x} \text{ and } x_k^* \to x^* \text{ with } x_k^* \in F(x_k), \\ \text{for } k = 1, 2, \dots\Big\}$$

If Ω is locally closed around $\bar{x} \in \Omega$, the cone

$$N(\bar{x};\Omega) = \operatorname*{Lim\,sup}_{x \xrightarrow{\Omega} \bar{x}} \widehat{N}(\bar{x};\Omega)$$

is said to be the *limiting* (or *basic/Mordukhovich*) *normal cone* to Ω at $\bar{x} \in \Omega$. If $\bar{x} \notin \Omega, N(\bar{x};\Omega) = \emptyset$ by convention.

The graph of a multifunction $\Phi : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is defined by

$$gph\Phi := \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m : y \in \Phi(x)\}.$$

For every $(\bar{x}, \bar{y}) \in gph\Phi$, we call the multifunction $\widehat{D}^*\Phi : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$,

$$\widehat{D}^*\Phi(\bar{x}, \bar{y})(y^*) := \Big\{x^* \in \mathbb{R}^n : (x^*, -y^*) \in \widehat{N}\big((\bar{x}, \bar{y}); gph\Phi\big)\Big\} \quad \forall y^* \in \mathbb{R}^m$$

the Fréchet coderivative of Φ at $(\bar{x}, \bar{y})$. The multifunction $D^*\Phi(\bar{x}, \bar{y})$ given by setting

$$D^*\Phi(\bar{x}, \bar{y})(y^*) := \Big\{x^* \in \mathbb{R}^n : (x^*, -y^*) \in N\big((\bar{x}, \bar{y}); gph\Phi\big)\Big\} \quad \forall y^* \in \mathbb{R}^m$$

is called the *Mordukhovich* (or *limiting/normal*) *coderivative* of Φ at $(\bar{x}, \bar{y})$. One says that Φ is graphically regular at $(\bar{x}, \bar{y}) \in gph\Phi$ if

$$\widehat{D}^*\Phi(\bar{x}, \bar{y})(y^*) = D^*\Phi(\bar{x}, \bar{y})(y^*).$$

The last condition can be written equivalently as

$$\widehat{N}((\bar{x}, \bar{y}); gph\Phi) = N((\bar{x}, \bar{y}); gph\Phi).$$

The feasible region of the problem $(ET_1(\bar{w}))$ is rewritten as follows

$$\mathcal{F}(r,b) := \{x \in \mathbb{R}^n : \|x\| \le r, a^T x + b \le 0\},$$

which depends on the parameter (r,b).

Denote by

$$N(x;\mathcal{F}(r,b)) := \{v \in \mathbb{R}^n : \langle v, y - x\rangle \le 0 \quad \forall y \in \mathcal{F}(r,b)\}$$

the normal cone to the convex set $\mathcal{F}(r,b)$ at x.

It is easy to see that

$$N(x;\mathcal{F}(r,b)) = \begin{cases} \{0\} & \text{if } \|x\| < r, a^T x + b < 0, \\ \{\theta x : \theta \ge 0\} & \text{if } \|x\| = r, a^T x + b < 0, \\ \{\gamma a : \gamma \ge 0\} & \text{if } \|x\| < r, a^T x + b = 0, \\ \{\theta x + \gamma a : \theta \ge 0, \gamma \ge 0\} & \text{if } \|x\| = r, a^T x + b = 0, \\ \emptyset & \text{if } \|x\| > r \text{ or } a^T x + b > 0. \end{cases} \tag{4.23}$$

For every $(x,r,b) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}$, we put

$$\mathcal{N}(x,r,b) = N(x;\mathcal{F}(r,b)).$$

If $r \le 0$ then it is convenient to set $\mathcal{N}(x,r,b) = \emptyset$ for all $x \in \mathbb{R}^n$. Hence $\mathcal{N} : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \rightrightarrows \mathbb{R}^n$ is a multifunction with closed convex values and is called *the normal cone mapping related to parametric* $(ET_1(\bar{w}))$.

Computing coderivatives of the normal cone mapping of a system of inequalities plays an important role in sensitivity and stability analysis of parameterized optimization and equilibrium problems. This research started in the 90s with the paper [29], where the authors obtained an exact formula for the Mordukhovich normal cone in the case when the given set is a convex polyhedron and then developed by Henrion et al. [45] and Ban et al. [9]. Recently, many authors have studied coderivatives of the normal cone mapping of polyhedral convex sets under linear and nonlinear perturbations (see, for instance, [9, 29, 45–47, 73, 74, 83–86]). In [63] and [85], the coderivatives of the normal cone mapping of the Euclidean ball with perturbed radius were estimated. Meanwhile, the researchers started to attack a more difficult case, when the given set is defined by many nonlinear inequalities (see [42] and references therein).

Recently, many authors have used coderivative tools to characterize the Lipschitzian stability of LCQP problems and of TRSs, which are two special subclasses of the QCQP problems (see [63, 85]).

In this section, we calculate and estimate the Fréchet and Mordukhovich coderivatives of the normal cone mapping related to the parametric $(ET_1(\bar{w}))$.

Fréchet coderivative of $\mathcal{N}(\cdot)$

Fix $\bar{\omega} := (\bar{x}, \bar{r}, \bar{b}, \bar{v}) \in gph\mathcal{N}$, we compute and estimate the Fréchet coderivative of the normal cone mapping. Before stating the main result, we consider the following lemmas.

Lemma 4.2. *The assertions are valid:*

(a) *If* $\|\bar{x}\| < \bar{r}$ *and* $a^T\bar{x} + \bar{b} < 0$, *then* $\bar{v} = 0$ *and*

$$\widehat{D}^*\mathcal{N}(\bar{\omega})(v^*) = \{(0_{\mathbb{R}^n}, 0_{\mathbb{R}}, 0_{\mathbb{R}})\};$$

(b) *If* $\|\bar{x}\| = \bar{r}, a^T\bar{x} + \bar{b} < 0,$ *and* $\bar{v} = \theta\bar{x}$ *with* $\theta > 0$ *then*

$$\widehat{D}^*\mathcal{N}(\bar{\omega})(v^*) = \begin{cases} \Omega_1(\bar{\omega})(v^*) & \text{if } \langle v^*, \bar{x}\rangle = 0, \\ \emptyset & \text{if } \langle v^*, \bar{x}\rangle \neq 0; \end{cases}$$

(c) *If* $\|\bar{x}\| = \bar{r}, a^T\bar{x} + \bar{b} < 0,$ *and* $\bar{v} = 0$ *then*

$$\widehat{D}^*\mathcal{N}(\bar{\omega})(v^*) = \begin{cases} \Omega_2(\bar{\omega})(v^*) & \text{if } \langle v^*, \bar{x}\rangle \geq 0, \\ \emptyset & \text{if } \langle v^*, \bar{x}\rangle < 0; \end{cases}$$

(d) *If* $\|\bar{x}\| < \bar{r}, a^T\bar{x} + \bar{b} = 0,$ *and* $\bar{v} = \gamma a$ *with* $\gamma > 0$ *then*

$$\widehat{D}^*\mathcal{N}(\bar{\omega})(v^*) = \begin{cases} \Omega_3(\bar{\omega})(v^*) & \text{if } \langle v^*, a\rangle = 0, \\ \emptyset & \text{if } \langle v^*, a\rangle \neq 0; \end{cases}$$

(e) *If* $\|\bar{x}\| < \bar{r}, a^T\bar{x} + \bar{b} = 0,$ *and* $\bar{v} = 0$ *then*

$$\widehat{D}^*\mathcal{N}(\bar{\omega})(v^*) = \begin{cases} \Omega_4(\bar{\omega})(v^*) & \text{if } \langle v^*, a\rangle \geq 0, \\ \emptyset & \text{if } \langle v^*, a\rangle < 0; \end{cases}$$

where

$$\Omega_1(\bar{\omega})(v^*) := \{(x^*, r^*, b^*) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} : b^* = 0,\ x^* = -\frac{r^*}{\bar{r}}\bar{x} + \theta v^*\},$$
$$\Omega_2(\bar{\omega})(v^*) := \{(x^*, r^*, b^*) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} : b^* = 0,\ r^* \leq 0,\ x^* = -\frac{r^*}{\bar{r}}\bar{x}\},$$
$$\Omega_3(\bar{\omega})(v^*) := \{(x^*, r^*, b^*) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} : r^* = 0,\ x^* = b^*a\},$$
$$\Omega_4(\bar{\omega})(v^*) := \{(x^*, r^*, b^*) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} : r^* = 0,\ x^* = b^*a,\ b^* \geq 0\}.$$

Proof. Put:

$$\begin{aligned} \mathcal{F}_1(r) &:= \{x \in \mathbb{R}^n : \|x\| \leq r\}, \\ \mathcal{F}_2(b) &:= \{x \in \mathbb{R}^n : a^Tx + b \leq 0\}, \\ \mathcal{N}_1(x, r) &:= N(x; \mathcal{F}_1(r)), \\ \mathcal{N}_2(x, b) &:= N(x; \mathcal{F}_2(b)). \end{aligned}$$

If $a^T\bar{x} + \bar{b} < 0$, then $\mathcal{N}(\bar{\omega}) = \mathcal{N}_1(\bar{x}, \bar{r})$. Since $\mathcal{N}_1(\cdot)$ does not depend on $\bar{b}$, we have

$$\widehat{D}^*\mathcal{N}(\bar{\omega})(v^*) = \{(x^*, r^*, b^*) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} : b^* = 0,$$
$$(x^*, r^*) \in \widehat{D}^*\mathcal{N}_1(\bar{x}, \bar{r}, \bar{v})(v^*)\}.$$

Similarly, if $\|\bar{x}\| < \bar{r}$, then $\mathcal{N}(\bar{\omega}) = \mathcal{N}_2(\bar{x}, \bar{r})$. Since $\mathcal{N}_2(\cdot)$ does not depend on r, we obtain

$$\widehat{D}^*\mathcal{N}(\bar{\omega})(v^*) = \{(x^*, r^*, b^*) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} : r^* = 0,$$
$$(x^*, b^*) \in \widehat{D}^*\mathcal{N}_2(\bar{x}, \bar{b}, \bar{v})(v^*)\}.$$

Applying [86, Theorem 3.2] to $\mathcal{F}_1(r)$ and $\mathcal{F}_2(b)$, we deduce immediately the desired results. $\square$

Lemma 4.3. *The following assertions hold:*

(i) *If* $\|\bar{x}\| = \bar{r}, a^T\bar{x} + \bar{b} = 0$ *and* $\bar{v} = \theta\bar{x} + \gamma a, \theta > 0, \gamma > 0,$ *then*

$$\widehat{D}^*\mathcal{N}(\bar{\omega})(v^*) \subset \begin{cases} \Omega_5(\bar{\omega})(v^*) & \text{if } \langle v^*, \bar{x}\rangle = 0 \text{ and } \langle v^*, a\rangle = 0, \\ \emptyset & \text{otherwise,} \end{cases}$$

where

$$\Omega_5(\bar{\omega})(v^*) := \{(x^*, r^*, b^*) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} : \langle x^*, \bar{x}\rangle + r^*\bar{r} + b^*\bar{b} = 0\}.$$

(ii) *If* $\|\bar{x}\| = \bar{r}, a^T\bar{x} + \bar{b} = 0$ *and* $\bar{v} = \theta\bar{x}$ *with* $\theta > 0,$ *then*

$$\widehat{D}^*\mathcal{N}(\bar{\omega})(v^*) \subset \begin{cases} \Omega_5^1(\bar{\omega})(v^*) & \text{if } \langle v^*, \bar{x}\rangle = 0 \text{ and } \langle v^*, a\rangle \geq 0, \\ \emptyset & \text{otherwise} \end{cases}$$

where

$$\Omega_5^1(\bar{\omega})(v^*) := \{(x^*, r^*, b^*) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}_+ : \langle x^*, \bar{x}\rangle + r^*\bar{r} + b^*\bar{b} = 0\}.$$

(iii) *If* $\|\bar{x}\| = \bar{r}, a^T\bar{x} + \bar{b} = 0$ *and* $\bar{v} = \gamma a$ *with* $\gamma > 0,$ *then*

$$\widehat{D}^*\mathcal{N}(\bar{\omega})(v^*) \subset \begin{cases} \Omega_5^2(\bar{\omega})(v^*) & \text{if } \langle v^*, a\rangle = 0 \text{ and } \langle v^*, \bar{x}\rangle \geq 0, \\ \emptyset & \text{otherwise,} \end{cases}$$

where

$$\Omega_5^2(\bar{\omega})(v^*) := \{(x^*, r^*, b^*) \in \mathbb{R}^n \times \mathbb{R}_- \times \mathbb{R} : \langle x^*, \bar{x}\rangle + r^*\bar{r} + b^*\bar{b} = 0\}.$$

Proof. Let $(x^*, r^*, b^*) \in \widehat{D}^*\mathcal{N}(\bar{\omega})(v^*)$. This means that

$$\limsup_{(\widetilde{x},\widetilde{r},\widetilde{b},\widetilde{v})) \overset{gph\mathcal{N}}{\to} \bar{\omega}} \frac{\langle x^*, \widetilde{x} - \bar{x}\rangle + r^*(\widetilde{r} - \bar{r}) + b^*(\widetilde{b} - \bar{b}) - \langle v^*, \widetilde{v} - \bar{v}\rangle}{\|\widetilde{x} - \bar{x}\| + |\widetilde{r} - \bar{r}| + |\widetilde{b} - \bar{b}| + \|\widetilde{v} - \bar{v}\|} \leq 0. \quad (4.24)$$

Choose $\widetilde{r} \downarrow \bar{r}, \widetilde{x} = \frac{\widetilde{r}}{\bar{r}}\bar{x}$ and $\widetilde{b} = \frac{\widetilde{r}}{\bar{r}}\bar{b}$. Since $\|\widetilde{x}\| = \widetilde{r}$ and $a^T\widetilde{x} + \widetilde{b} = 0$, we choose $\widetilde{v} = \bar{v}$. From (4.24) it follows that

$$0 \geq \limsup_{\widetilde{r}\downarrow\bar{r}} \frac{\langle x^*, \frac{\widetilde{r}}{\bar{r}}\bar{x} - \bar{x}\rangle + r^*(\widetilde{r} - \bar{r}) + b^*(\frac{\widetilde{r}}{\bar{r}}\bar{b} - \bar{b})}{\|\frac{\widetilde{r}}{\bar{r}}\bar{x} - \bar{x}\| + |\widetilde{r} - \bar{r}| + |\frac{\widetilde{r}}{\bar{r}}\bar{b} - \bar{b}|} = \frac{\langle x^*, \bar{x}\rangle + r^*\bar{r} + b^*\bar{b}}{\|\bar{x}\| + |\bar{r}| + |\bar{b}|},$$

which gives $\langle x^*, \bar{x}\rangle + r^*\bar{r} + b^*\bar{b} \leq 0$.

Repeating the preceding arguments for the case where $\widetilde{r} \uparrow \bar{r}$, we get

$$\langle x^*, \bar{x}\rangle + r^*\bar{r} + b^*\bar{b} \geq 0.$$

From the last two inequalities, we have

$$\langle x^*, \bar{x}\rangle + r^*\bar{r} + b^*\bar{b} = 0. \tag{4.25}$$

Choose $\widetilde{x} = \bar{x}, \widetilde{b} = \bar{b}, \widetilde{r} = \bar{r}$, and $\widetilde{v} = \bar{v} + t\bar{v}$ for $t \in \mathbb{R}$. From (4.24),

$$0 \geq \limsup_{t\uparrow 0} \frac{-\langle v^*, t\bar{v}\rangle}{\|t\bar{v}\|} = \frac{\langle v^*, \bar{v}\rangle}{\|\bar{v}\|}$$

and

$$0 \geq \limsup_{t\downarrow 0} \frac{-\langle v^*, t\bar{v}\rangle}{\|t\bar{v}\|} = -\frac{\langle v^*, \bar{v}\rangle}{\|\bar{v}\|}.$$

Hence

$$\langle v^*, \bar{v}\rangle = 0. \tag{4.26}$$

(i) Let $\widetilde{x} = \bar{x}, \widetilde{b} = \bar{b}, \widetilde{r} = \bar{r}$, and $\widetilde{v} = \bar{v} + t\bar{x}, t > 0$. Then, (4.24) gives

$$\langle v^*, \bar{x}\rangle \geq 0. \tag{4.27}$$

Choose $\widetilde{x} = \bar{x}, \widetilde{b} = \bar{b}, \widetilde{r} = \bar{r}$ and $\widetilde{v} = \bar{v} + ta, t > 0$. According to (4.24),

$$\langle v^*, a\rangle \geq 0. \tag{4.28}$$

By (4.26), (4.27), and (4.28), we obtain that $\langle v^*, \bar{x}\rangle = 0$ and $\langle v^*, a\rangle = 0$.

(ii) From (4.26) it follows $\langle v^*, \bar{x}\rangle = 0$.

Choose $\widetilde{x} = \bar{x}, \widetilde{b} = \bar{b}, \widetilde{r} = \bar{r}$, and $\widetilde{v} = \bar{v} + ta$ with $t \downarrow 0$. Then, (4.24) yields

$$0 \geq \limsup_{t \downarrow 0} \frac{-\langle v^*, ta \rangle}{\|ta\|} = -\frac{\langle v^*, a \rangle}{\|a\|}.$$

This leads to $\langle v^*, a \rangle \geq 0$.

Choose $\widetilde{x} = \bar{x}, \widetilde{r} = \bar{r}$, $\widetilde{b} \uparrow \bar{b}$ and $\widetilde{v} = \bar{v} = \theta \bar{x}$. From (4.24) it follows

$$0 \geq \limsup_{\widetilde{b} \uparrow \bar{b}} \frac{b^*(\widetilde{b} - \bar{b})}{|\widetilde{b} - \bar{b}|} = -b^*.$$

Hence $b^* \geq 0$.

(iii) From (4.26), we have $\langle v^*, a \rangle = 0$.

Choose $\widetilde{x} = \bar{x}, \widetilde{b} = \bar{b}, \widetilde{r} = \bar{r}$, and $\widetilde{v} = \bar{v} + t\bar{x}$ with $t \downarrow 0$. From (4.24) one has

$$0 \geq \limsup_{t \downarrow 0} \frac{-\langle v^*, t\bar{x} \rangle}{\|t\bar{x}\|} = -\frac{\langle v^*, \bar{x} \rangle}{\|\bar{x}\|},$$

which implies $\langle v^*, \bar{x} \rangle \geq 0$.

Choose $\widetilde{x} = \bar{x}, \widetilde{r} \downarrow \bar{r}$, $\widetilde{b} = \bar{b}$ and $\widetilde{v} = \bar{v} = \gamma a$. Then, (4.24) gives

$$0 \geq \limsup_{\widetilde{r} \downarrow \bar{r}} \frac{r^*(\widetilde{r} - \bar{r})}{|\widetilde{r} - \bar{r}|} = r^*.$$

The proof is complete. $\square$

Denote $pos\{\bar{x}, a\} := \{\theta \bar{x} + \gamma a : \theta \geq 0, \gamma \geq 0\}$.

Lemma 4.4. *If* $\|\bar{x}\| = \bar{r}, a^T \bar{x} + \bar{b} = 0$ *and* $\bar{v} = 0$, *then*

$$\widehat{D}^* \mathcal{N}(\bar{\omega})(v^*) \subset \begin{cases} \Omega_6(\bar{\omega})(v^*) & \text{if } \langle v^*, \bar{x} \rangle \geq 0 \text{ and } \langle v^*, a \rangle \geq 0, \\ \emptyset & \text{otherwise,} \end{cases}$$

where

$$\Omega_6(\bar{\omega})(v^*) := \{(x^*, r^*, b^*) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} : \langle x^*, \bar{x} \rangle + r^* \bar{r} + b^* \bar{b} = 0,$$
$$x^* \in pos\{\bar{x}, a\},\ b^* \geq 0,\ r^* \leq 0\}.$$

Proof. Let $(x^*, r^*, b^*) \in \widehat{D}^*\mathcal{N}(\bar{\omega})(v^*)$. Then, (4.24) holds.

Choose $\widetilde{x} = \bar{x}, \widetilde{b} = \bar{b}, \widetilde{r} = \bar{r}$ and $\widetilde{v} = t\bar{x}$ with $t \downarrow 0$. According to (4.24),

$$0 \geq \limsup_{t \downarrow 0} \frac{-\langle v^*, t\bar{x}\rangle}{\|t\bar{x}\|} = -\frac{\langle v^*, \bar{x}\rangle}{\|\bar{x}\|}.$$

This implies $\langle v^*, \bar{x}\rangle \geq 0$.

We now choose $\widetilde{x} = \bar{x}, \widetilde{b} = \bar{b}, \widetilde{r} = \bar{r}$ and $\widetilde{v} = ta$ with $t \downarrow 0$. Then, (4.24) becomes

$$0 \geq \limsup_{t \downarrow 0} \frac{-\langle v^*, ta\rangle}{\|ta\|} = -\frac{\langle v^*, a\rangle}{\|a\|},$$

that is, $\langle v^*, a\rangle \geq 0$.

Choose $\widetilde{r} \downarrow \bar{r}, \widetilde{x} = \frac{\widetilde{r}}{\bar{r}}\bar{x}, \widetilde{b} = \frac{\widetilde{r}}{\bar{r}}\bar{b}$ and $\tilde{v} = \bar{v} = 0$. Then, one has (4.25).

Next, choose $\widetilde{x} = \bar{x}, \widetilde{r} \downarrow \bar{r}, \widetilde{b} = \bar{b}$ and $\widetilde{v} = \bar{v} = 0$. From (4.24),

$$0 \geq \limsup_{\widetilde{r} \downarrow \bar{r}} \frac{r^*(\widetilde{r} - \bar{r})}{|\widetilde{r} - \bar{r}|} = r^*.$$

Choose $\widetilde{x} = \bar{x}, \widetilde{r} = \bar{r}, \widetilde{b} \uparrow \bar{b}$ and $\widetilde{v} = 0$. Then, (4.24) yields

$$0 \geq \limsup_{\widetilde{b} \uparrow \bar{b}} \frac{b^*(\widetilde{b} - \bar{b})}{|\widetilde{b} - \bar{b}|} = -b^*,$$

which means $b^* \geq 0$.

Finally, choose $\widetilde{r} = \bar{r}, \widetilde{b} = \bar{b}, \widetilde{x} \xrightarrow{\partial\mathcal{F}(\bar{r},\bar{b})} \bar{x}$ and $\widetilde{v} = \bar{v} = 0$. By (4.24),

$$\limsup_{\widetilde{x} \xrightarrow{\partial\mathcal{F}(\bar{r},\bar{b})} \bar{x}} \frac{\langle x^*, \widetilde{x} - \bar{x}\rangle}{\|\widetilde{x} - \bar{x}\|} \leq 0. \tag{4.29}$$

Let any $\widetilde{x}_k \xrightarrow{\partial\mathcal{F}(\bar{r},\bar{b})} \bar{x}$ such that

$$\lim_{k \to \infty} \frac{\widetilde{x}_k - \bar{x}}{\|\widetilde{x}_k - \bar{x}\|} = u.$$

Then, $u \in T(\bar{x}, \partial\mathcal{F}(\bar{r}, \bar{b}))$. By (4.29), we have $\langle x^*, u\rangle \leq 0$ for every $u \in T(\bar{x}, \partial\mathcal{F}(\bar{r}, \bar{b}))$. This gives $x^* \in pos\{\bar{x}, a\}$. This finishes the proof. $\square$

By Lemmas 4.2–4.4, we get the following main theorem of this section:

Theorem 4.6. *For every $\bar{\omega} = (\bar{x}, \bar{r}, \bar{b}, \bar{v}) \in gph\mathcal{N}$, the assertions are valid:*

(a) *If* $\|\bar{x}\| < \bar{r}$ *and* $a^T\bar{x} + \bar{b} < 0$, *then* $\bar{v} = 0$ *and*

$$\widehat{D}^*\mathcal{N}(\bar{\omega})(v^*) = \{(0_{\mathbb{R}^n}, 0_{\mathbb{R}}, 0_{\mathbb{R}})\};$$

(b) *If* $\|\bar{x}\| = \bar{r}, a^T\bar{x} + \bar{b} < 0,$ *and* $\bar{v} = \theta\bar{x}$ *with* $\theta > 0$ *then*

$$\widehat{D}^*\mathcal{N}(\bar{\omega})(v^*) = \begin{cases} \Omega_1(\bar{\omega})(v^*) & \textit{if } \langle v^*, \bar{x}\rangle = 0, \\ \emptyset & \textit{if } \langle v^*, \bar{x}\rangle \neq 0; \end{cases}$$

(c) *If* $\|\bar{x}\| = \bar{r}, a^T\bar{x} + \bar{b} < 0,$ *and* $\bar{v} = 0$ *then*

$$\widehat{D}^*\mathcal{N}(\bar{\omega})(v^*) = \begin{cases} \Omega_2(\bar{\omega})(v^*) & \textit{if } \langle v^*, \bar{x}\rangle \geq 0, \\ \emptyset & \textit{if } \langle v^*, \bar{x}\rangle < 0; \end{cases}$$

(d) *If* $\|\bar{x}\| < \bar{r}, a^T\bar{x} + \bar{b} = 0,$ *and* $\bar{v} = \gamma a$ *with* $\gamma > 0$ *then*

$$\widehat{D}^*\mathcal{N}(\bar{\omega})(v^*) = \begin{cases} \Omega_3(\bar{\omega})(v^*) & \textit{if } \langle v^*, a\rangle = 0, \\ \emptyset & \textit{if } \langle v^*, a\rangle \neq 0; \end{cases}$$

(e) *If* $\|\bar{x}\| < \bar{r}, a^T\bar{x} + \bar{b} = 0,$ *and* $\bar{v} = 0$ *then*

$$\widehat{D}^*\mathcal{N}(\bar{\omega})(v^*) = \begin{cases} \Omega_4(\bar{\omega})(v^*) & \textit{if } \langle v^*, a\rangle \geq 0, \\ \emptyset & \textit{if } \langle v^*, a\rangle < 0; \end{cases}$$

(f) *If* $\|\bar{x}\| = \bar{r}, a^T\bar{x} + \bar{b} = 0,$ *and* $\bar{v} = \theta\bar{x} + \gamma a$ *with* $\theta > 0, \gamma > 0$ *then*

$$\widehat{D}^*\mathcal{N}(\bar{\omega})(v^*) \subset \begin{cases} \Omega_5(\bar{\omega})(v^*) \ \textit{if } \langle v^*, \bar{x}\rangle = 0 \ \textit{ and } \langle v^*, a\rangle = 0, \\ \emptyset \qquad\qquad\quad \textit{otherwise} \end{cases}$$

(g) *If* $\|\bar{x}\| = \bar{r}, a^T\bar{x} + \bar{b} = 0,$ *and* $\bar{v} = \theta\bar{x}$ *with* $\theta > 0,$ *then*

$$\widehat{D}^*\mathcal{N}(\bar{\omega})(v^*) \subset \begin{cases} \Omega_5^1(\bar{\omega})(v^*) & \text{if } \langle v^*, \bar{x}\rangle = 0 \text{ and } \langle v^*, a\rangle \geq 0, \\ \emptyset & \text{otherwise} \end{cases}$$

(h) *If* $\|\bar{x}\| = \bar{r}, a^T\bar{x} + \bar{b} = 0,$ *and* $\bar{v} = \gamma a$ *with* $\gamma > 0,$ *then*

$$\widehat{D}^*\mathcal{N}(\bar{\omega})(v^*) \subset \begin{cases} \Omega_5^2(\bar{\omega})(v^*) & \text{if } \langle v^*, \bar{x}\rangle = 0 \text{ and } \langle v^*, \bar{x}\rangle \geq 0, \\ \emptyset & \text{otherwise} \end{cases}$$

(i) *If* $\|\bar{x}\| = \bar{r}, a^T\bar{x} + \bar{b} = 0,$ *and* $\bar{v} = 0$ *then*

$$\widehat{D}^*\mathcal{N}(\bar{\omega})(v^*) \subset \begin{cases} \Omega_6(\bar{\omega})(v^*) & \text{if } \langle v^*, \bar{x}\rangle \geq 0, \quad \text{and } \langle v^*, a\rangle \geq 0, \\ \emptyset & \text{otherwise.} \end{cases}$$

Mordukhovich coderivative of $\mathcal{N}(\cdot)$

To estimate the Mordukhovich coderivative of $\mathcal{N}(\cdot)$, we consider some lemmas.

Lemma 4.5. *For every* $\bar{\omega} = (\bar{x}, \bar{r}, \bar{b}, \bar{v}) \in gph\mathcal{N}$, *the following assertions are valid:*

(a) *If* $\|\bar{x}\| < \bar{r}$ *and* $a^T\bar{x} + \bar{b} < 0,$ *then* $\bar{v} = 0$ *and*

$$D^*\mathcal{N}(\bar{\omega})(v^*) = \{(0_{\mathbb{R}^n}, 0_{\mathbb{R}}, 0_{\mathbb{R}})\};$$

(b) *If* $\|\bar{x}\| = \bar{r}, a^T\bar{x} + \bar{b} < 0,$ *and* $\bar{v} = \theta\bar{x},$ *with* $\theta > 0$ *then*

$$D^*\mathcal{N}(\bar{\omega})(v^*) = \begin{cases} \Omega_1(\bar{\omega})(v^*) & \text{if } \langle v^*, \bar{x}\rangle = 0, \\ \emptyset & \text{if } \langle v^*, \bar{x}\rangle \neq 0; \end{cases}$$

(c) *If* $\|\bar{x}\| = \bar{r}, a^T\bar{x} + \bar{b} < 0,$ *and* $\bar{v} = 0$ *then*

$$D^*\mathcal{N}(\bar{\omega})(v^*) = \begin{cases} \{0_{\mathbb{R}^{n+2}}\} & \text{if } \langle v^*, \bar{x}\rangle < 0, \\ \Omega_2(\bar{\omega})(v^*) & \text{if } \langle v^*, \bar{x}\rangle > 0, \\ \Omega_2'(\bar{\omega})(v^*) & \text{if } \langle v^*, \bar{x}\rangle = 0; \end{cases}$$

(d) *If* $\|\bar{x}\| < \bar{r}, a^T\bar{x} + \bar{b} = 0,$ *and* $\bar{v} = \gamma a$ *with* $\gamma > 0$ *then*

$$D^*\mathcal{N}(\bar{\omega})(v^*) = \begin{cases} \Omega_3(\bar{\omega})(v^*) & \text{if } \langle v^*, a\rangle = 0, \\ \emptyset & \text{if } \langle v^*, a\rangle \neq 0; \end{cases}$$

(e) *If* $\|\bar{x}\| < \bar{r}, a^T\bar{x} + \bar{b} = 0,$ *and* $\bar{v} = 0$ *then*

$$D^*\mathcal{N}(\bar{\omega})(v^*) = \begin{cases} \{(0_{\mathbb{R}^{n+2}})\} & \text{if } \langle v^*, a\rangle < 0, \\ \Omega_4(\bar{\omega})(v^*) & \text{if } \langle v^*, a\rangle > 0, \\ \Omega_3(\bar{\omega})(v^*) & \text{if } \langle v^*, a\rangle = 0, \end{cases}$$

where

$$\Omega_2'(\bar{\omega})(v^*) := \big\{(x^*, r^*, b^*) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} : b^* = 0,\ x^* = -\frac{r^*}{\bar{r}}\bar{x}\big\}.$$

Proof. Repeating the arguments in the proof of Lemma 4.2 and using [86, Theorem 3.3], we get the required conclusions. □

Lemma 4.6. *Assume that* $\|\bar{x}\| = \bar{r},\ a^T\bar{x} + \bar{b} = 0$ *and* $\bar{v} \neq 0$. *The following assertions hold:*

(i) *If* $\bar{v} = \theta\bar{x}$ *with* $\theta > 0$, *then*

$$D^*\mathcal{N}(\bar{\omega})(v^*) \subset \begin{cases} \Omega_5(\bar{\omega})(v^*) \cup \Omega_1(\bar{\omega})(v^*) & \text{if } \langle v^*, \bar{x}\rangle = 0, \\ \emptyset & \text{if } \langle v^*, \bar{x}\rangle \neq 0; \end{cases}$$

(ii) *If* $\bar{v} = \gamma a$ *with* $\gamma > 0$, *then*

$$D^*\mathcal{N}(\bar{\omega})(v^*) \subset \begin{cases} \Omega_5(\bar{\omega})(v^*) \cup \Omega_3(\bar{\omega})(v^*) & \text{if } \langle v^*, a\rangle = 0, \\ \emptyset & \text{if } \langle v^*, a\rangle \neq 0; \end{cases}$$

(iii) *If* $\bar{v} = \theta\bar{x} + \gamma a$ *with* $\theta > 0$ *and* $\gamma > 0$, *then*

$$D^*\mathcal{N}(\bar{\omega})(v^*) \subset \begin{cases} \Omega_5(\bar{\omega})(v^*) & \text{if } \langle v^*, \bar{v}\rangle = 0, \\ \emptyset & \text{if } \langle v^*, \bar{v}\rangle \neq 0. \end{cases}$$

Proof. For any $(x^*, r^*, b^*) \in D^*\mathcal{N}(\bar{\omega})(v^*)$, there exist $\omega_k = (x_k, r_k, b_k, v_k)$ satisfying $\omega_k \xrightarrow{gph\mathcal{N}} \bar{\omega} = (\bar{x}, \bar{r}, \bar{b}, \bar{v})$ and $(x_k^*, r_k^*, b_k^*, v_k^*) \to (x^*, r^*, b^*, v^*)$ such that

$$(x_k^*, r_k^*, b_k^*, v_k^*) \in \widehat{D}^*\mathcal{N}(\omega_k)(v_k^*). \tag{4.30}$$

From $\bar{v} \neq 0$ it follows $v_k \neq 0$ for every k. We distinguish the following two cases:

Case 1: $\|x_k\| = r_k$ for every k large enough. Then, we may assume that $\|x_k\| = r_k$ for every k. We next consider the following two subcases:

Subcase 1.1: $a^T x_k + b_k = 0$ for every k large enough. Then, we can assume that $a^T x_k + b_k = 0$ for all k. By Lemma 4.3, we have $(x_k^*, r_k^*, b_k^*) \in \Omega_5(\omega_k)(v_k^*)$, that is,

$$\langle x_k^*, x_k \rangle + r_k^* r_k + b_k^* b_k = 0, \langle v_k^*, x_k \rangle = 0 \text{ and } \langle v_k^*, a \rangle = 0.$$

Letting $k \to \infty$, one has

$$\langle x^*, \bar{x} \rangle + r^* \bar{r} + b^* \bar{b} = 0, \langle v^*, \bar{x} \rangle = 0 \text{ and } \langle v^*, a \rangle = 0.$$

This leads to $(x^*, r^*, b^*) \in \Omega_5(\bar{\omega})(v^*)$. Hence

$$D^*\mathcal{N}(\bar{\omega})(v^*) \subset \Omega_5(\bar{\omega})(v^*).$$

Subcase 1.2: there exists $\{k_l\} \subset \{k\}$ such that $a^T x_{k_l} + b_{k_l} < 0$ for all l. Then, $v_{k_l} = \theta_{k_l} x_{k_l}$ with $0 < \theta_{k_l} \to \theta$. By Lemma 4.2, we have $(x_{k_l}^*, r_{k_l}^*, b_{k_l}^*) \in \Omega_1(p_{k_l})(v_{k_l}^*)$, that is,

$$b_{k_l}^* = 0, x_{k_l}^* = -\frac{r_{k_l}^*}{r_{k_l}} x_{k_l} + \theta_{k_l} v_{k_l}^* \text{ and } \langle v_{k_l}^*, x_{k_l} \rangle = 0.$$

Letting $l \to \infty$, one gets

$$b^* = 0, x^* = -\frac{r^*}{\bar{r}} \bar{x} + \theta v^* \text{ and } \langle v^*, \bar{x} \rangle = 0.$$

This means $(x^*, r^*, b^*) \in \Omega_1(\bar{\omega})(v^*)$. Thus

$$D^*\mathcal{N}(\bar{\omega})(v^*) \subset \Omega_1(\bar{\omega})(v^*).$$

Case 2: there exists $\{k_s\} \subset \{k\}$ *such that* $\|x_{k_s}\| < r_{k_s}$. Since $v_{k_s} \neq 0$, $a^T x_{k_s} + b_{k_s} = 0$ for every s. Then, $v_{k_s} = \gamma_{k_s} a$ with $0 < \gamma_{k_s} \to \gamma$. By Lemma 4.2, we have $(x^*_{k_s}, r^*_{k_s},\ b^*_{k_s}) \in \Omega_3(p_{k_s})(v^*_{k_s})$, that is,

$$r^*_{k_s} = 0,\ x^*_{k_s} = b^*_{k_s} a \text{ and } \langle v^*_{k_s}, x_{k_s}\rangle = 0.$$

Passing the latter to limits as $s \to \infty$, we have

$$r^* = 0,\ x^* = b^* a \text{ and } \langle v^*, a\rangle = 0.$$

Hence $(x^*, r^*, b^*) \in \Omega_3(\bar{\omega})(v^*)$, and

$$D^*\mathcal{N}(\bar{\omega})(v^*) \subset \Omega_3(\bar{\omega})(v^*).$$

By the above arguments, we now prove (i), (ii) and (iii).

(i) If $\bar{v} = \theta\bar{x}$ with $\theta > 0$, then Case 2 does not occur. Hence

$$D^*\mathcal{N}(\bar{\omega})(v^*) \subset \begin{cases} \Omega_5(\bar{\omega})(v^*) \cup \Omega_1(\bar{\omega})(v^*) & \text{if } \langle v^*, \bar{x}\rangle = 0, \\ \emptyset & \text{if } \langle v^*, \bar{x}\rangle \neq 0. \end{cases}$$

(ii) If $\bar{v} = \gamma a$ with $\gamma > 0$, then Subcase 1.2 does not occur. Thus

$$D^*\mathcal{N}(\bar{\omega})(v^*) \subset \begin{cases} \Omega_5(\bar{\omega})(v^*) \cup \Omega_3(\bar{\omega})(v^*) & \text{if } \langle v^*, a\rangle = 0, \\ \emptyset & \text{if } \langle v^*, a\rangle \neq 0. \end{cases}$$

(iii) If $\bar{v} = \theta\bar{x} + \gamma a$ with $\theta > 0$ and $\gamma > 0$, then both Cases 1.2 and 2 do not occur. Therefore

$$D^*\mathcal{N}(\bar{\omega})(v^*) \subset \begin{cases} \Omega_5(\bar{\omega})(v^*) & \text{if } \langle v^*, \bar{v}\rangle = 0, \\ \emptyset & \text{if } \langle v^*, \bar{v}\rangle \neq 0. \end{cases}$$

The proof is complete. $\square$

Lemma 4.7. *If* $\|\bar{x}\| = \bar{r}, a^T\bar{x} + \bar{b} = 0$ *and* $\bar{v} = 0$, *then:*

$$D^*\mathcal{N}(\bar{\omega})(v^*) = \begin{cases} \{(0_{\mathbb{R}^n}, 0_{\mathbb{R}}, 0_{\mathbb{R}})\} & \text{if } \langle v^*, \bar{x}\rangle < 0 \;\; \text{and } \langle v^*, a\rangle < 0, \\ \Omega_4(\bar{\omega})(v^*) & \text{if } \langle v^*, \bar{x}\rangle < 0 \;\; \text{and } \langle v^*, a\rangle > 0, \\ \Omega_2(\bar{\omega})(v^*) & \text{if } \langle v^*, \bar{x}\rangle > 0 \;\; \text{and } \langle v^*, a\rangle < 0, \\ \Omega_2'(\bar{\omega})(v^*) & \text{if } \langle v^*, \bar{x}\rangle = 0 \;\; \text{and } \langle v^*, a\rangle < 0, \\ \Omega_3(\bar{\omega})(v^*) & \text{if } \langle v^*, \bar{x}\rangle < 0 \;\; \text{and } \langle v^*, a\rangle = 0; \end{cases}$$

and

$$D^*\mathcal{N}(\bar{\omega})(v^*) \subset \begin{cases} \Omega_7(\bar{\omega})(v^*) & \text{if } \langle v^*, \bar{x}\rangle > 0 \;\; \text{and } \langle v^*, a\rangle > 0, \\ \Omega_8(\bar{\omega})(v^*) & \text{if } \langle v^*, \bar{x}\rangle = 0 \;\; \text{and } \langle v^*, a\rangle > 0, \\ \Omega_9(\bar{\omega})(v^*) & \text{if } \langle v^*, \bar{x}\rangle > 0 \;\; \text{and } \langle v^*, a\rangle = 0, \\ \Omega_{10}(\bar{\omega})(v^*) & \text{if } \langle v^*, \bar{x}\rangle = 0 \;\; \text{and } \langle v^*, a\rangle = 0; \end{cases}$$

where

$$\Omega_7(\bar{\omega}) = \Omega_2(\bar{\omega})(v^*) \cup \Omega_4(\bar{\omega})(v^*) \cup \Omega_6(\bar{\omega})(v^*),$$

$$\Omega_8(\bar{\omega}) = \Omega_2'(\bar{\omega})(v^*) \cup \Omega_4(\bar{\omega})(v^*) \cup \Omega_5^1(\bar{\omega})(v^*) \cup \Omega_6(\bar{\omega})(v^*),$$

$$\Omega_9(\bar{\omega}) = \Omega_2(\bar{\omega})(v^*) \cup \Omega_3(\bar{\omega})(v^*) \cup \Omega_5^2(\bar{\omega})(v^*) \cup \Omega_6(\bar{\omega})(v^*),$$

$$\Omega_{10}(\bar{\omega}) = \Omega_2'(\bar{\omega})(v^*) \cup \Omega_3(\bar{\omega})(v^*) \cup \Omega_5(\bar{\omega})(v^*) \cup \Omega_6(\bar{\omega})(v^*).$$

Proof. We consider the following nine cases:

Case 1: $\langle v^*, \bar{x}\rangle < 0$ *and* $\langle v^*, a\rangle < 0$. Let any $(x^*, r^*, b^*) \in D^*\mathcal{N}(\bar{\omega})(v^*)$. Then, (4.30) holds. Since $\langle v^*, \bar{x}\rangle < 0$ and $\langle v^*, a\rangle < 0$, we may assume that $\langle v_k^*, x_k\rangle < 0$ and $\langle v_k^*, a\rangle < 0$ for every k. Fix any k.

If $\|x_k\| = r_k$ and $v_k \neq 0$, then $\widehat{D}^*\mathcal{N}(\omega_k)(v_k^*) = \emptyset$, by Lemmas 4.2 and 4.3. If $\|x_k\| = r_k$ and $v_k = 0$ then, by Lemmas 4.2 and 4.4, $\widehat{D}^*\mathcal{N}(\omega_k)(v_k^*) = \emptyset$. From Lemmas 4.2 and 4.3, $\widehat{D}^*\mathcal{N}(\omega_k)(v_k^*) = \emptyset$ if $a^T x_k + b_k = 0$ and $v_k \neq 0$. If $a^T x_k + b_k = 0$ and $v_k = 0$ then, by Lemmas 4.2 and 4.4, $\widehat{D}^*\mathcal{N}(\omega_k)(v_k^*) = \emptyset$. Hence $\widehat{D}^*\mathcal{N}(\omega_k)(v_k^*) \neq \emptyset$ if $\|x_k\| < r_k$ and $a^T x_k +$

$b_k < 0$. From Lemma 4.2 it follows $\widehat{D}^*\mathcal{N}(\omega_k)(v_k^*) = \{(0_{\mathbb{R}^n}, 0_{\mathbb{R}}, 0_{\mathbb{R}})\}$. This gives $x_k^* = 0,\ r_k^* = 0$ and $b_k^* = 0$. Letting $k \to \infty$, one has $x^* = 0, r^* = 0$ and $b^* = 0$. Hence $D^*\mathcal{N}(\bar{\omega})(v^*) \subset \{(0_{\mathbb{R}^n}, 0_{\mathbb{R}}, 0_{\mathbb{R}})\}$.

Conversely, let $r_k = \bar{r}, b_k = (1 - k^{-1})\bar{b} - (k^2)^{-1}, x_k = (1 - k^{-1})\bar{x}$ and $v_k = 0$. Then, $\|x_k\| < r_k,\ a^T x_k + b_k = -(k^2)^{-1} < 0$ and $v_k = 0$. Let $x_k^* = 0, r_k^* = 0, b_k^* = 0$. Then, we have (4.30) by Lemma 4.2. Hence $\{(0_{\mathbb{R}^n}, 0_{\mathbb{R}}, 0_{\mathbb{R}})\} \subset D^*\mathcal{N}(\bar{\omega})(v^*)$. The first conclusion is proved.

Case 2: $\langle v^*, \bar{x}\rangle < 0$ *and* $\langle v^*, a\rangle > 0$. For any $(x^*, r^*, b^*) \in D^*\mathcal{N}(\bar{\omega})(v^*)$, we have (4.30). Since $\langle v^*, \bar{x}\rangle < 0$ and $\langle v^*, a\rangle > 0$, we may assume that $\langle v^*, x_k\rangle < 0$ and $\langle v_k^*, a\rangle > 0$ for every k. Fix any k. From Lemmas 4.2 and 4.3, if $\|x_k\| = r_k$ and $v_k \neq 0$ then $\widehat{D}^*\mathcal{N}(\omega_k)(v_k^*) = \emptyset$. If $\|x_k\| = r_k$ and $v_k = 0$ then, by Lemmas 4.2 and 4.4, $\widehat{D}^*\mathcal{N}(\omega_k)(v_k^*) = \emptyset$.

Therefore, in order to get that $\widehat{D}^*\mathcal{N}(\omega_k)(v_k^*) \neq \emptyset$, we must have $\|x_k\| < r_k$. Consider the following two subcases:

Subcase 2.1: $a^T x_k + b_k = 0$. By Lemma 4.2, if $v_k \neq 0$ then $\widehat{D}^*\mathcal{N}(\omega_k)(v_k^*) = \emptyset$. If $v_k = 0$ then we have $(x_k^*, r_k^*, b_k^*) \in \Omega_4(\omega_k)(v_k^*)$, that is,

$$r_k^* = 0, x_k^* = b_k^* a,\ b_k^* \geq 0 \text{ and } \langle v_k^*, a\rangle \geq 0.$$

Passing to the limits as $k \to \infty$, we have

$$r^* = 0, x^* = b^* a,\ b^* \geq 0 \text{ and } \langle v^*, a\rangle \geq 0,$$

which mean $(x^*, r^*, b^*) \in \Omega_4(\bar{\omega})(v^*)$.

Subcase 2.2: $a^T x_k + b_k < 0$. Then $\widehat{D}^*\mathcal{N}(\omega_k)(v_k^*) = \{0_{\mathbb{R}^{n+2}}\}$ from Lemma 4.2. It implies $x_k^* = 0, r_k^* = 0$ and $b_k^* = 0$. Letting $k \to \infty$, one has $x^* = 0, r^* = 0$ and $b^* = 0$. Hence $D^*\mathcal{N}(\bar{\omega})(v^*) \subset \{(0_{\mathbb{R}^n}, 0_{\mathbb{R}}, 0_{\mathbb{R}})\}$.

By Subcases 2.1 and 2.2, we have $D^*\mathcal{N}(\bar{\omega})(v^*) \subset \Omega_4(\bar{\omega})(v^*)$.

Conversely, for any $(x^*, r^*, b^*) \in \Omega_4(\bar{\omega})(v^*)$, we obtain that $r^* = 0$, $b^* \geq 0$ and $\langle v^*, a\rangle \geq 0$. Choose $r_k = \bar{r}, b_k = (1 - k^{-1})\bar{b},\ x_k = (1 - k^{-1})\bar{x}$. Then, $\|x_k\| < r_k,\ a^T x_k + b_k = 0$ and $v_k = 0$. We choose $r_k^* = 0$,

$b_k^* = b^*, x_k^* = b_k^* a$ and $v_k^* = v^*$. By Lemma 4.2, we have (4.30). Hence $(x^*, r^*, b^*) \in D^*\mathcal{N}(\bar{\omega})(v^*)$. Then, we get the assertion (ii).

Case 3: $\langle v^*, \bar{x}\rangle > 0$ and $\langle v^*, a\rangle < 0$. Let $(x^*, r^*, b^*) \in D^*\mathcal{N}(\bar{\omega})(v^*)$. Then, (4.30) holds. Since $\langle v^*, \bar{x}\rangle > 0$ and $\langle v^*, a\rangle < 0$, we may assume that $\langle v^*, x_k\rangle > 0$ and $\langle v_k^*, a\rangle < 0$ for every k. Fix any k. If $a^T x_k + b_k = 0$ and $v_k \neq 0$ then $\widehat{D}^*\mathcal{N}(\omega_k)(v_k^*) = \emptyset$ by Lemmas 4.2 and 4.3. If $a^T x_k + b_k = 0$ and $v_k = 0$ then, by Lemmas 4.2 and 4.4, $\widehat{D}^*\mathcal{N}(\omega_k)(v_k^*) = \emptyset$. Consequently, to get $\widehat{D}^*\mathcal{N}(\omega_k)(v_k^*) \neq \emptyset$, we must have $a^T x_k + b_k < 0$. We now consider the following two subcases:

Subcase 3.1: $\|x_k\| = r_k$. From Lemma 4.2, $\widehat{D}^*\mathcal{N}(\omega_k)(v_k^*) = \emptyset$ if $v_k \neq 0$. If $v_k = 0$ then, by Lemma 4.2, we have $(x_k^*, r_k^*, b_k^*) \in \Omega_2(\omega_k)(v_k^*)$, that is,

$$b_k^* = 0, x_k^* = -\frac{r_k^*}{r_k} x_k, r_k^* \leq 0 \text{ and } \langle v_k^*, x_k\rangle \geq 0.$$

Letting $k \to \infty$, we obtain $b^* = 0,\ x^* = -\frac{r^*}{\bar{r}}\bar{x},\ r^* \leq 0$ and $\langle v^*, \bar{x}\rangle = 0$, which imply $(x^*, r^*, b^*) \in \Omega_2(\bar{\omega})(v^*)$.

Subcase 3.2: $a^T x_k + b_k < 0$. We have $\widehat{D}^*\mathcal{N}(\omega_k)(v_k^*) = \{0_{\mathbb{R}^{n+2}}\}$ by Lemma 4.2, i.e., $x_k^* = 0, r_k^* = 0$ and $b_k^* = 0$. Passing the latter to limits as $k \to \infty$, one has $x^* = 0, r^* = 0$ and $b^* = 0$. By Subcases 3.1 and 3.2, we have $D^*\mathcal{N}(\bar{\omega})(v^*) \subset \Omega_2(\bar{\omega})(v^*)$.

Conversely, let any $(x^*, r^*, b^*) \in \Omega_2(\bar{\omega})(v^*)$, i.e., $b^* = 0$, $x^* = -\frac{r^*}{\bar{r}}\bar{x}$, $r^* \leq 0$ and $\langle v^*, \bar{x}\rangle = 0$. Choose $r_k = (1-k^{-1})\bar{r}$, $b_k = (1-k^{-1})\bar{b} - (k^2)^{-1}$, $x_k = (1-k^{-1})\bar{x}$. Then, $\|x_k\| = r_k$, $a^T x_k + b_k < 0$ and $v_k = 0$. Let $r_k^* = r^*$, $b_k^* = b^*$, $x_k^* = -\frac{r_k^*}{r_k} x_k$ and $v_k^* = v^*$. From Lemma 4.2, we get (4.30). This gives $(x^*, r^*, b^*) \in D^*\mathcal{N}(\bar{\omega})(v^*)$. The assertion (iii) is shown.

Case 4: $\langle v^*, \bar{x}\rangle = 0$ and $\langle v^*, a\rangle < 0$. For any $(x^*, r^*, b^*) \in D^*\mathcal{N}(\bar{\omega})(v^*)$, we get (4.30). Since $\langle v^*, a\rangle < 0$, we can assume that $\langle v_k^*, a\rangle < 0$ for every k. Fix any k. By Lemmas 4.2 and 4.3, if $a^T x_k + b_k = 0$ and $v_k \neq 0$ then $\widehat{D}^*\mathcal{N}(\omega_k)(v_k^*) = \emptyset$. If $a^T x_k + b_k = 0$ and $v_k = 0$ then, by Lemmas 4.2 and 4.4, $\widehat{D}^*\mathcal{N}(\omega_k)(v_k^*) = \emptyset$. Consequently, to get $\widehat{D}^*\mathcal{N}(\omega_k)(v_k^*) \neq \emptyset$, we must have $a^T x_k + b_k < 0$. Consider the following three subcases:

Subcase 4.1: $\|x_k\| = r_k$ and $v_k = 0$. To obtain $\widehat{D}^*\mathcal{N}(\omega_k)(v_k^*) \neq \emptyset$, by Lemma 4.2, we must have $\langle v_k^*, x_k\rangle \geq 0$. Then,

$$b_k^* = 0,\ x_k^* = -\frac{r_k^*}{r_k} x_k, r_k^* \leq 0 \text{ and } \langle v_k^*, x_k\rangle \geq 0.$$

Passing the latter to limits as $k \to \infty$, we obtain

$$b^* = 0,\ x^* = -\frac{r^*}{\bar{r}}\bar{x},\ r^* \leq 0 \text{ and } \langle v^*, \bar{x}\rangle \geq 0.$$

Hence $(x^*, r^*, b^*) \in \Omega_2(\bar{\omega})(v^*) \subset \Omega_2'(\bar{\omega})(v^*)$.

Subcase 4.2: $\|x_k\| = r_k$ and $v_k \neq 0$. This implies $v_k = \theta_k x_k$ with $\theta_k = (\|x_k\|^{-1}\|v_k\|) \downarrow 0$. To obtain that $\widehat{D}^*\mathcal{N}(\omega_k)(v_k^*) \neq \emptyset$, by Lemma 4.2, we must have $\langle v_k^*, x_k\rangle = 0$. Then,

$$b_k^* = 0, x_k^* = -\frac{r_k^*}{r_k} x_k + \theta_k v_k^* \text{ and } \langle v_k^*, x_k\rangle = 0.$$

Letting $k \to \infty$, we get $b^* = 0, x^* = -\frac{r^*}{\bar{r}}\bar{x}$ and $\langle v^*, \bar{x}\rangle = 0$. This leads to $(x^*, r^*, b^*) \in \Omega_2'(\bar{\omega})(v^*)$.

Subcase 4.3: $\|x_k\| < r_k$. By Lemma 4.2, $\widehat{D}^*\mathcal{N}(\omega_k)(v_k^*) = \{(0_{\mathbb{R}^{n+2}})\}$, i.e., $x_k^* = 0, r_k^* = 0$ and $b_k^* = 0$. Letting $k \to \infty$ yields $x^* = 0$, $r^* = 0$ and $b^* = 0$. Thus $(x^*, r^*, b^*) \in \Omega_2'(\bar{\omega})(v^*)$.

Conversely, we let any $(x^*, r^*, b^*) \in \Omega_2'(\bar{\omega})(v^*)$, that is, $b^* = 0$, $x^* = -\frac{r^*}{\bar{r}}\bar{x}$ and $\langle v^*, \bar{x}\rangle = 0$. Choose $r_k = \bar{r},\ x_k = \bar{x},\ b_k = \bar{b} - k^{-1}$. Then, $\|x_k\| = r_k,\ a^T x_k + b_k = -k^{-1} < 0$ and $v_k = \theta_k x_k$ with $\theta_k \downarrow 0$. Let $r_k^* = r^*, b_k^* = b^*, x_k^* = -\frac{r_k^*}{r_k} x_k + \theta_k v_k^*$ and $v_k^* = v^*$. Then, we obtain (4.30) by Lemma 4.2, which follows $(x^*, r^*, b^*) \in D^*\mathcal{N}(\bar{\omega})(v^*)$. This gives the assertion (iv).

Case 5: $\langle v^*, \bar{x}\rangle < 0$ and $\langle v^*, a\rangle = 0$. Let $(x^*, r^*, b^*) \in D^*\mathcal{N}(\bar{\omega})(v^*)$. Then, (4.30) holds. Since $\langle v^*, \bar{x}\rangle < 0$, we may assume that $\langle v_k^*, x_k\rangle < 0$ for every k. Fix any k. If $\|x_k\| = r_k$ and $v_k \neq 0$ then, by Lemmas 4.2

and 4.3, $\widehat{D}^*\mathcal{N}(\omega_k)(v_k^*) = \emptyset$. If $\|x_k\| = r_k$ and $v_k = 0$ then, by Lemmas 4.2 and 4.4, $\widehat{D}^*\mathcal{N}(\omega_k)(v_k^*) = \emptyset$. To get $\widehat{D}^*\mathcal{N}(\omega_k)(v_k^*) \neq \emptyset$, we must have $\|x_k\| < r_k$. Consider the following three subcases:

Subcase 5.1: $a^T x_k + b_k = 0$ and $v_k = 0$. To get $\widehat{D}^*\mathcal{N}(\omega_k)(v_k^*) \neq \emptyset$, by Lemma 4.2, we must have $\langle v_k^*, a\rangle \geq 0$. This gives

$$r_k^* = 0, x_k^* = b_k^* a,\ b_k^* \geq 0 \text{ and } \langle v_k^*, a\rangle \geq 0.$$

Passing to limits as $k \to \infty$ yields

$$r^* = 0, x^* = b^* a,\ b^* \geq 0 \text{ and } \langle v^*, a\rangle \geq 0,$$

which means $(x^*, r^*, b^*) \in \Omega_4(\bar{\omega})(v^*) \subset \Omega_3(\bar{\omega})(v^*)$.

Subcase 5.2: $a^T x_k + b_k = 0$ and $v_k \neq 0$. This implies $v_k = \theta_k a$ with $\theta_k = (\|a\|^{-1}\|v_k\|) \downarrow 0$. To get that $\widehat{D}^*\mathcal{N}(\omega_k)(v_k^*) \neq \emptyset$, by Lemma 4.2, we must have $\langle v_k^*, a\rangle = 0$. Then,

$$r_k^* = 0, x_k^* = b_k^* a \text{ and } \langle v_k^*, a\rangle = 0.$$

Letting $k \to \infty$,

$$r^* = 0, x^* = b^* a \text{ and } \langle v^*, a\rangle \geq 0.$$

Hence $(x^*, r^*, b^*) \in \Omega_3(\bar{\omega})(v^*)$.

Subcase 5.3: $a^T x_k + b_k < 0$. By Lemma 4.2, it follows that $\widehat{D}^*\mathcal{N}(\omega_k)(v_k^*) = \{(0_{\mathbb{R}^n}, 0_{\mathbb{R}}, 0_{\mathbb{R}})\}$, that is, $x_k^* = 0, r_k^* = 0$ and $b_k^* = 0$. Letting $k \to \infty$, one has $x^* = 0, r^* = 0$ and $b^* = 0$, which gives $(x^*, r^*, b^*) \in \Omega_3(\bar{\omega})(v^*)$.

Conversely, for any $(x^*, r^*, b^*) \in \Omega_3(\bar{\omega})(v^*)$, we obtain that $r^* = 0$, $x^* = b^* a$ and $\langle v^*, a\rangle \geq 0$. Choose $r_k = \bar{r}$, $x_k = (1-k^{-1})\bar{x}$, $b_k = (1-k^{-1})\bar{b}$. Then, $\|x_k\| < r_k$, $a^T x_k + b_k = 0$ and $v_k = \gamma_k a$ with $\gamma_k \downarrow 0$. Let $r_k^* = r^*$, $b_k^* = b^*, x_k^* = b_k^* a$ and $v_k^* = v^*$. Then, we have (4.30) by Lemma 4.2. Hence $(x^*, r^*, b^*) \in D^*\mathcal{N}(\bar{\omega})(v^*)$. The assertion (v) follows.

Case 6: $\langle v^*, \bar{x}\rangle > 0$ and $\langle v^*, a\rangle > 0$. For $(x^*, r^*, b^*) \in D^*\mathcal{N}(\bar{\omega})(v^*)$, (4.30) follows. Since $\langle v^*, \bar{x}\rangle > 0$ and $\langle v^*, a\rangle > 0$, we may assume that

$\langle v_k^*, x_k\rangle > 0$ and $\langle v_k^*, a\rangle > 0$ for every k. Fix any k. If $v_k \neq 0$ then, by Lemmas 4.2 and 4.3, $\widehat{D}^*\mathcal{N}(\omega_k)(v_k^*) = \emptyset$. To get $\widehat{D}^*\mathcal{N}(\omega_k)(v_k^*) \neq \emptyset$, we must have $v_k = 0$. Hence $(x_k^*, r_k^*, b_k^*) \in \Omega_2(\omega_k)(v_k^*) \cup \Omega_4(\omega_k)(v_k^*) \cup \Omega_6(\omega_k)(v_k^*)$ by Lemmas 4.2 and 4.4. This follows

$$(x^*, r^*, b^*) \in \Omega_2(\bar\omega)(v^*) \cup \Omega_4(\bar\omega)(v^*) \cup \Omega_6(\bar\omega)(v^*),$$

which leads to the assertion (vi).

Case 7: $\langle v^*, \bar x\rangle = 0$ and $\langle v^*, a\rangle > 0$. For $(x^*, r^*, b^*) \in D^*\mathcal{N}(\bar\omega)(v^*)$, (4.30) holds. Since $\langle v^*, a\rangle > 0$, we may assume that $\langle v_k^*, a\rangle > 0$ for every k. Fix any k. By Lemmas 4.2–4.4, we have $(x_k^*, r_k^*, b_k^*) \in \Omega_1(\omega_k)(v_k^*) \cup \Omega_2(\omega_k)(v_k^*) \cup \Omega_4(\omega_k)(v_k^*) \cup \Omega_5^1(\omega_k)(v_k^*) \cup \Omega_6(\omega_k)(v_k^*)$. This gives

$$(x^*, r^*, b^*) \in \Omega_2'(\bar\omega)(v^*) \cup \Omega_4(\bar\omega)(v^*) \cup \Omega_5^1(\bar\omega)(v^*) \cup \Omega_6(\bar\omega)(v^*).$$

The assertion (vii) is proved.

Case 8: $\langle v^*, \bar x\rangle > 0$ and $\langle v^*, a\rangle = 0$. For $(x^*, r^*, b^*) \in D^*\mathcal{N}(\bar\omega)(v^*)$, one gets (4.30). Since $\langle v^*, \bar x\rangle > 0$, we may assume that $\langle v_k^*, x_k\rangle > 0$ for every $k \geq 1$. Fix any k. From Lemmas 4.2–4.4 it follows that $(x_k^*, r_k^*, b_k^*) \in \Omega_2(\omega_k)(v_k^*) \cup \Omega_3(\omega_k)(v_k^*) \cup \Omega_5^2(\omega_k)(v_k^*) \cup \Omega_6(\omega_k)(v_k^*)$. Letting $k \to \infty$, one has

$$(x^*, r^*, b^*) \in \Omega_2(\bar\omega)(v^*) \cup \Omega_3(\bar\omega)(v^*) \cup \Omega_5^2(\bar\omega)(v^*) \cup \Omega_6(\bar\omega)(v^*).$$

The assertion (viii) is proved.

Case 9: $\langle v^*, \bar x\rangle = 0$ and $\langle v^*, a\rangle = 0$. For $(x^*, r^*, b^*) \in D^*\mathcal{N}(\bar\omega)(v^*)$, (4.30) holds. Fix any k. By Lemmas 4.2–4.4, we have

$$(x_k^*, r_k^*, b_k^*) \in \big(\Omega_1(\omega_k) \cup \Omega_2(\omega_k) \cup \Omega_3(\omega_k) \cup \Omega_4(\omega_k) \cup \Omega_5(\omega_k) \cup \Omega_6(\omega_k)\big)(v_k^*).$$

Passing the latter to limits as $k \to \infty$ yields

$$(x^*, r^*, b^*) \in \Omega_2'(\bar\omega)(v^*) \cup \Omega_3(\bar\omega)(v^*) \cup \Omega_5(\bar\omega)(v^*) \cup \Omega_6(\bar\omega)(v^*).$$

The conclusion of the assertion (ix) is shown. The proof is then complete. $\square$

By the above arguments, we get the main result in this subsection as follows.

Theorem 4.7. *For every $\bar{\omega} = (\bar{x}, \bar{r}, \bar{b}, \bar{v}) \in gph\mathcal{N}$, the assertions are valid:*

(a) *If $\|\bar{x}\| < \bar{r}$ and $a^T\bar{x} + \bar{b} < 0$, then $\bar{v} = 0$ and*

$$D^*\mathcal{N}(\bar{\omega})(v^*) = \{(0_{\mathbb{R}^n}, 0_{\mathbb{R}}, 0_{\mathbb{R}})\};$$

(b) *If $\|\bar{x}\| = \bar{r}, a^T\bar{x} + \bar{b} < 0$, and $\bar{v} = \theta\bar{x}$ with $\theta > 0$ then*

$$D^*\mathcal{N}(\bar{\omega})(v^*) = \begin{cases} \Omega_1(\bar{\omega})(v^*) & \text{if } \langle v^*, \bar{x}\rangle = 0, \\ \emptyset & \text{if } \langle v^*, \bar{x}\rangle \neq 0; \end{cases}$$

(c) *If $\|\bar{x}\| = \bar{r}, a^T\bar{x} + \bar{b} < 0$, and $\bar{v} = 0$ then*

$$D^*\mathcal{N}(\bar{\omega})(v^*) = \begin{cases} \{0_{\mathbb{R}^{n+2}}\} & \text{if } \langle v^*, \bar{x}\rangle < 0, \\ \Omega_2(\bar{\omega})(v^*) & \text{if } \langle v^*, \bar{x}\rangle > 0, \\ \Omega_2'(\bar{\omega})(v^*) & \text{if } \langle v^*, \bar{x}\rangle = 0; \end{cases}$$

(d) *If $\|\bar{x}\| < \bar{r}, a^T\bar{x} + \bar{b} = 0$, and $\bar{v} = \gamma a$ with $\gamma > 0$ then*

$$D^*\mathcal{N}(\bar{\omega})(v^*) = \begin{cases} \Omega_3(\bar{\omega})(v^*) & \text{if } \langle v^*, a\rangle = 0, \\ \emptyset & \text{if } \langle v^*, a\rangle \neq 0; \end{cases}$$

(e) *If $\|\bar{x}\| < \bar{r}, a^T\bar{x} + \bar{b} = 0$, and $\bar{v} = 0$ then*

$$D^*\mathcal{N}(\bar{\omega})(v^*) = \begin{cases} \{0_{\mathbb{R}^{n+2}}\} & \text{if } \langle v^*, a\rangle < 0, \\ \Omega_4(\bar{\omega})(v^*) & \text{if } \langle v^*, a\rangle > 0, \\ \Omega_3(\bar{\omega})(v^*) & \text{if } \langle v^*, a\rangle = 0; \end{cases}$$

(f) *If $\|\bar{x}\| = \bar{r}, a^T\bar{x} + \bar{b} = 0$, and $\bar{v} = \theta\bar{x} + \gamma a$ with $\theta > 0, \gamma > 0$ then*

$$D^*\mathcal{N}(\bar{\omega})(v^*) \subset \begin{cases} \Omega_5(\bar{\omega})(v^*) & \text{if } \langle v^*, \bar{v}\rangle = 0, \\ \emptyset & \text{if } \langle v^*, \bar{v}\rangle \neq 0; \end{cases}$$

(g) *If* $\|\bar{x}\| = \bar{r}, a^T\bar{x} + \bar{b} = 0,$ *and* $\bar{v} = \theta\bar{x}$ *with* $\theta > 0$ *then*

$$D^*\mathcal{N}(\bar{\omega})(v^*) \subset \begin{cases} \Omega_5(\bar{\omega})(v^*) \cup \Omega_1(\bar{\omega})(v^*) & \textit{if } \langle v^*, \bar{x}\rangle = 0, \\ \emptyset & \textit{if } \langle v^*, \bar{x}\rangle \neq 0; \end{cases}$$

(h) *If* $\|\bar{x}\| = \bar{r}, a^T\bar{x} + \bar{b} = 0,$ *and* $\bar{v} = \gamma a$ *with* $\gamma > 0$ *then*

$$D^*\mathcal{N}(\bar{\omega})(v^*) \subset \begin{cases} \Omega_5(\bar{\omega})(v^*) \cup \Omega_3(\bar{\omega})(v^*) & \textit{if } \langle v^*, a\rangle = 0, \\ \emptyset & \textit{if } \langle v^*, a\rangle \neq 0; \end{cases}$$

(i) *If* $\|\bar{x}\| = \bar{r},\ a^T\bar{x} + \bar{b} = 0$ *and* $\bar{v} = 0$ *then*

$$D^*\mathcal{N}(\bar{\omega})(v^*) = \begin{cases} \{(0_{\mathbb{R}^n}, 0_{\mathbb{R}}, 0_{\mathbb{R}})\} & \textit{if } \langle v^*, \bar{x}\rangle < 0 \ \textit{ and } \langle v^*, a\rangle < 0, \\ \Omega_4(\bar{\omega})(v^*) & \textit{if } \langle v^*, \bar{x}\rangle < 0 \ \textit{ and } \langle v^*, a\rangle > 0, \\ \Omega_2(\bar{\omega})(v^*) & \textit{if } \langle v^*, \bar{x}\rangle > 0 \ \textit{ and } \langle v^*, a\rangle < 0, \\ \Omega_2'(\bar{\omega})(v^*) & \textit{if } \langle v^*, \bar{x}\rangle = 0 \ \textit{ and } \langle v^*, a\rangle < 0, \\ \Omega_3(\bar{\omega})(v^*) & \textit{if } \langle v^*, \bar{x}\rangle < 0 \ \textit{ and } \langle v^*, a\rangle = 0; \end{cases}$$

and

$$D^*\mathcal{N}(\bar{\omega})(v^*) \subset \begin{cases} \Omega_7(\bar{\omega})(v^*) & \textit{if } \langle v^*, \bar{x}\rangle > 0 \ \textit{ and } \langle v^*, a\rangle > 0, \\ \Omega_8(\bar{\omega})(v^*) & \textit{if } \langle v^*, \bar{x}\rangle = 0 \ \textit{ and } \langle v^*, a\rangle > 0, \\ \Omega_9(\bar{\omega})(v^*) & \textit{if } \langle v^*, \bar{x}\rangle > 0 \ \textit{ and } \langle v^*, a\rangle = 0, \\ \Omega_{10}(\bar{\omega})(v^*) & \textit{if } \langle v^*, \bar{x}\rangle = 0 \ \textit{ and } \langle v^*, a\rangle = 0. \end{cases}$$

4.3.3. Lipschitzian stability

In this subsection, we use obtained results and the Mordukhovich criterion (see [73, Theorem 4.10]) for the local Lipschitz-like property of multifunctions to investigate Lipschitzian stability of $(ET_1(\bar{\omega}))$ with respect to the linear perturbations. We always assume that $(ET_1(\bar{w}))$ satisfies LICQ.

The stationary solution set of $(ET_1(w))$ is rewritten by $S(Q,q,r,b)$. Recall that (see, for instance, [32, Proposition 1.3.4]), under LICQ, x is a stationary solution of $(ET_1(\bar{w}))$ if and only if

$$\langle Qx+q, y-x\rangle \geq 0 \quad \forall y \in \mathcal{F}(r,b),$$

i.e., x is a global optimal solution of the generalized equation

$$0 \in Qx + q + N(x;\mathcal{F}(r,b)). \tag{4.31}$$

We can rewrite (4.31) as follows

$$y \in H(x,z) + M(x,z), \tag{4.32}$$

where $y := -q, z := (Q,r,b), H(x,z) := Qx$ and $M(x,z) := \mathcal{N}(x,r,b)$.

Denote by $\mathbb{R}^{n\times n}_s$ the linear subspace of symmetric $n\times n$ matrices in $\mathbb{R}^{n\times n}$ and put $Z := \mathbb{R}^{n\times n}_s \times \mathbb{R} \times \mathbb{R}$. Then, $S(\cdot)$ can be interpreted as the multifunction $\widetilde{S} : Z \times \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ defined by

$$\widetilde{S}(z,y) = \{x \in \mathbb{R}^n : y \in H(x,z) + M(x,z)\}. \tag{4.33}$$

Then, we have

$$\widetilde{S}(z,y) = S(Q,q,r,b).$$

The following lemma is used to prove the main theorem.

Lemma 4.8. *The set $gph\mathcal{N}$ is closed in $\mathbb{P} := \mathbb{R}^n \times (0,+\infty) \times \mathbb{R} \times \mathbb{R}^n$.*

Proof. Suppose that $\omega_k = (x_k, r_k, b_k, v_k) \xrightarrow{gph\mathcal{N}} \bar{\omega} = (\bar{x}, \bar{r}, \bar{b}, \bar{v}) \in \mathbb{P}$. We now prove $\bar{\omega} \in gph\mathcal{N}$, that is, $\bar{v} \in N(\bar{x};\mathcal{F}(\bar{r},\bar{b}))$. Indeed, we consider the following four cases:

Case 1: $\|\bar{x}\| < \bar{r}$ and $a^T\bar{x} + \bar{b} < 0$. For every k large enough, $\|x_k\| < r_k$, $a^T x_k + b_k < 0$ and $v_k = 0$. It follows $\bar{v} = 0$. Therefore $\bar{v} \in N(\bar{x};\mathcal{F}(\bar{r},\bar{b})) = \{0\}$.

Case 2: $\|\bar{x}\| = \bar{r}$ and $a^T\bar{x} + \bar{b} < 0$. Then, $N(\bar{x};\mathcal{F}(\bar{r},\bar{b})) = \{\theta\bar{x}, \theta \geq 0\}$. For every k large enough, we have $a^T x_k + b_k < 0$. Fix such a index k. Consider the following subcases:

Subcase 2.1: $\|x_k\| < r_k$. Then, $v_k = 0$, and $\bar{v} = 0 \in N(\bar{x}; \mathcal{F}(\bar{r}, \bar{b}))$.

Subcase 2.2: $\|x_k\| = r_k$. Then, $v_k = \theta_k x_k$ with

$$0 < \theta_k = \|x_k\|^{-1}.\|v_k\| \to \bar{\theta} := \|\bar{x}\|^{-1}.\|\bar{v}\|.$$

This yields $\bar{v} = \bar{\theta}\bar{x} \in N(\bar{x}; \mathcal{F}(\bar{r}, \bar{b}))$.

Case 3: $\|\bar{x}\| < \bar{r}$ *and* $a^T\bar{x} + \bar{b} = 0$. Then, $N(\bar{x}; \mathcal{F}(\bar{r}, \bar{b})) = \{\gamma a, \gamma \geq 0\}$. For every k large enough, we have $\|x_k\| < r_k$. Fix such a index k. Consider the following subcases:

Subcase 3.1: $a^T x_k + b_k < 0$. Then $v_k = 0$ and $\bar{v} = 0 \in N(\bar{x}; \mathcal{F}(\bar{r}, \bar{b}))$.

Subcase 3.2: $a^T x_k + b_k = 0$. In this case, we obtain that $v_k = \gamma_k a$, where $0 < \gamma_k = \|a\|^{-1}.\|v_k\| \to \bar{\gamma}$ with $\bar{\gamma} := \|a\|^{-1}.\|\bar{v}\|$. It follows that $\bar{v} = \bar{\gamma} a \in N(\bar{x}; \mathcal{F}(\bar{r}, \bar{b}))$.

Case 4: $\|\bar{x}\| = \bar{r}$ *and* $a^T\bar{x} + \bar{b} = 0$. Then, $N(\bar{x}; \mathcal{F}(\bar{r}, \bar{b})) = pos\{\bar{x}, a\}$. Fix any k. Consider the following four subcases:

Subcase 4.1: $\|x_k\| < r_k, a^T x_k + b_k < 0$. Then, $v_k = 0$. This gives that $\bar{v} = 0 \in N(\bar{x}; \mathcal{F}(\bar{r}, \bar{b}))$.

Subcase 4.2: $\|x_k\| = r_k, a^T x_k + b_k < 0$. Then, $v_k = \theta_k x_k$ with $0 < \theta_k = \|x_k\|^{-1}.\|v_k\| \to \bar{\theta} := \|\bar{x}\|^{-1}.\|\bar{v}\|$ and $\bar{v} = \bar{\theta}\bar{x} \in N(\bar{x}; \mathcal{F}(\bar{r}, \bar{b}))$.

Subcase 4.3: $\|x_k\| < r_k, a^T x_k + b_k = 0$. Then, $v_k = \gamma_k a$ with $0 < \gamma_k = \|a\|^{-1}.\|v_k\| \to \bar{\gamma} := \|a\|^{-1}.\|\bar{v}\|$. Thus $\bar{v} = \bar{\gamma} a \in N(\bar{x}; \mathcal{F}(\bar{r}, \bar{b}))$.

Subcase 4.4: $\|x_k\| = r_k, a^T x_k + b_k = 0$. Then, $v_k = \theta_k x_k + \gamma_k a$ with $\theta_k \geq 0$ and $\gamma_k \geq 0$.

If $\|\gamma_k\| < +\infty$ then we can assume that $\gamma_k \to \bar{\gamma} \geq 0$. One has

$$\theta_k = \frac{\|v_k - \gamma_k a\|}{\|x_k\|} \to \bar{\theta} := \frac{\|\bar{v} - \bar{\gamma} a\|}{\|\bar{x}\|} \geq 0.$$

Thus $\bar{v} = \bar{\theta}\bar{x} + \bar{\gamma} a \in N(\bar{x}; \mathcal{F}(\bar{r}, \bar{b}))$.

If $\|\gamma_k\| \to +\infty$ then

$$\frac{v_k}{\gamma_k} = \frac{\theta_k}{\gamma_k} x_k + a. \tag{4.34}$$

If $\{\theta_k/\gamma_k\}$ is bounded then we can assume that $\theta_k/\gamma_k \to \mu$. From (4.34) it follows $0 = \mu\bar{x}+a$, contrary to the fact that $(ET_1(\bar{w}))$ satisfies (LICQ). Otherwise, if $\|\theta_k/\gamma_k\| \to +\infty$ then (4.34) gives

$$\frac{v_k}{\gamma_k} : \frac{\theta_k}{\gamma_k} = x_k + \left(\frac{\theta_k}{\gamma_k}\right)^{-1} a.$$

Letting $k \to \infty$ yields $0 = \bar{x}$. This contradicts the fact that $\|\bar{x}\| = \bar{r} > 0$. The lemma is proved. □

The following theorem estimates the Mordukhovich coderivative of $\widetilde{S}(\cdot)$.

Theorem 4.8. *Consider the problem $(ET_1(\bar{w}))$ and $(\bar{z}, \bar{y}, \bar{x}) \in gph\widetilde{S}$. For each $x^* \in \mathbb{R}^n$, if $(y^*, z^*) \in D^*\widetilde{S}(\bar{z}, \bar{y}, \bar{x})(x^*)$ then:*

$$\begin{aligned}
&\bar{Q}y^* = 2x^*,\\
&Q^*_{ij} = -y^*_i\bar{x}_j,\\
&(x^*, r^*, b^*) \in D^*\mathcal{N}(\bar{x}, \bar{r}, \bar{b}, \bar{v})(-y^*);
\end{aligned}$$

where $\bar{z} = (\bar{Q}, \bar{r}, \bar{b})$, $\bar{v} = \bar{y} - H(\bar{x}, \bar{z}) = -\bar{q} - \bar{Q}\bar{x}$, $z^* = (Q^*, r^*, b^*)$ and Q^*_{ij} is the (i, j)th element of Q^*.

Proof. By Lemma 4.8, we have $\mathcal{N}$ is locally closed around $(\bar{x}, \bar{r}, \bar{b}) \in gph\mathcal{N}$; hence M is locally closed around $(\bar{x}, \bar{z}) \in gphM$.

With similar analysis the proof of [63, Lemmas 4.1–4.3], we obtain that

$$D^*M(\bar{x}, \bar{z}, \bar{v})(v^*) = \{(x^*, 0_{\mathbb{R}^{n\times n}_s}, r^*, b^*) : (x^*, r^*, b^*) \in D^*\mathcal{N}(\bar{\omega})(v^*)\} \tag{4.35}$$

and

$$\nabla H(\bar{x}, \bar{z})^*(v^*) = \{\bar{Q}v^*\} \times (v^*_i\bar{x}_j) \times \{0_{\mathbb{R}}\}, \tag{4.36}$$

where $(v^*_i\bar{x}_j)$ is the $n \times n$ matrix whose (i, j)th element is $v^*_i\bar{x}_j$.

From [62, Theorem 4.3] it follows that

$$D^*\widetilde{S}(\bar{z}, \bar{y}, \bar{x})(x^*) \subset \Omega_{H,\bar{y}}(x^*),$$

where

$$\Omega_{H,\bar{y}}(x^*) = \bigcup_{v^* \in \mathbb{R}^n} \Big\{(z^*, y^*) \in Z^* \times \mathbb{R}^n : (-x^*, z^*, y^*) \in \nabla H(\bar{x}, \bar{z})^*(v^*) \\ \times \{-v^*\} + D^*M(\bar{x}, \bar{z}, \bar{v})(v^*) \times \{0_{\mathbb{R}^n}\}\Big\}.$$

For each $x^* \in \mathbb{R}^n$, if $(y^*, z^*) \in D^*\widetilde{S}(\bar{z}, \bar{y}, \bar{x})(x^*)$ then $(y^*, z^*) \in \Omega_{H,\bar{y}}(x^*)$, that is,

$$\begin{cases} -y^* = v^*, \\ -x^* = \bar{Q}v^* + x^*, \\ Q^*_{ij} = v^*_i \bar{x}_j, \\ (x^*, r^*, b^*) \in D^*\mathcal{N}(\bar{x}, \bar{r}, \bar{b}, \bar{v})(v^*). \end{cases}$$

The latter system is equivalent to

$$\begin{cases} \bar{Q}y^* = 2x^*, \\ Q^*_{ij} = -y^*_i \bar{x}_j, \\ (x^*, r^*, b^*) \in D^*\mathcal{N}(\bar{x}, \bar{r}, \bar{b}, \bar{v})(-y^*). \end{cases}$$

This establishes the desired formula. □

The Mordukhovich criterion (see [73, Theorem 4.10]) for the local Lipschitz-like property of multifunctions shows that $\widetilde{S}(\cdot)$ is locally Lipschitz-like around $(\bar{z}, \bar{y}, \bar{x}) \in gph\widetilde{S}$ if and only if

$$D^*\widetilde{S}(\bar{z}, \bar{y}, \bar{x})(0) = \{0\}. \tag{4.37}$$

Since $D^*\widetilde{S}(\bar{z}, \bar{y}, \bar{x})(0) = \{0\}$ is equivalent to $D^*S(\bar{z}, \bar{y}, \bar{x})(0) = \{0\}$, we conclude that $\widetilde{S}(\cdot)$ is locally Lipschitz-like around $(\bar{z}, \bar{y}, \bar{x}) \in gph\widetilde{S}$ if and only if S is locally Lipschitz-like around $(\bar{Q}, \bar{q}, \bar{r}, \bar{b}, \bar{x}) \in gphS$.

From Theorem (4.8) it follows that (4.37) holds if the following system

$$\begin{cases} \bar{Q}y^* = 0, \\ Q^*_{ij} = -y^*_i \bar{x}_j, \\ (0, r^*, b^*) \in D^*\mathcal{N}(\bar{\omega})(-y^*), \end{cases} \tag{4.38}$$

has a unique solution $(Q^*, r^*, b^*, y^*) = 0$, which is equivalent to that

$$\begin{cases} \bar{Q}y^* = 0, \\ (0, r^*, b^*) \in D^*\mathcal{N}(\bar{\omega})(-y^*), \end{cases} \tag{4.39}$$

has a unique solution $(r^*, b^*, y^*) = 0$. If $det\bar{Q} \neq 0$ then (4.39) reduces to that

$$(0, r^*, b^*) \in D^*\mathcal{N}(\bar{\omega})(0), \tag{4.40}$$

has a unique solution $(r^*, b^*) = 0$.

The following theorem shows some sufficient conditions for the local Lipschitz-like property of $S(\cdot)$.

Theorem 4.9. *The multifunction* $(\widetilde{Q}, \widetilde{q}, \widetilde{r}, \widetilde{b}) \mapsto S(\widetilde{Q}, \widetilde{q}, \widetilde{r}, \widetilde{b})$ *is locally Lipschitz-like around* $(\bar{Q}, \bar{q}, \bar{r}, \bar{b}, \bar{x}) \in gphS$ *if at least one of the following conditions is satisfied:*

(i) $\|\bar{x}\| < \bar{r}, a^T\bar{x} + \bar{b} < 0$ *and* $det\bar{Q} \neq 0$*;*

(ii) $\|\bar{x}\| = \bar{r}, a^T\bar{x} + \bar{b} < 0$ *and* $\bar{Q}\bar{x} + \bar{q} = \theta\bar{x}, \theta > 0$;

(iii) $\|\bar{x}\| = \bar{r}, a^T\bar{x} + \bar{b} < 0$, $\bar{Q}\bar{x} + \bar{q} = 0$, $rank(\bar{Q}; \bar{x}) = n$ *and* $\langle\bar{x}, u\rangle = 0$ *for every* $u \in Null(\bar{Q})$*;*

(iv) $\|\bar{x}\| < \bar{r}, a^T\bar{x} + \bar{b} = 0$, $\bar{Q}\bar{x} + \bar{q} = \gamma a, \gamma > 0$, *and* $rank(\bar{Q}; a) = n$*;*

(v) $\|\bar{x}\| < \bar{r}, a^T\bar{x} + \bar{b} = 0$, $\bar{Q}\bar{x} + \bar{q} = 0$, $rank(\bar{Q}; a) = n$ *and* $\langle a, u\rangle = 0$ *for every* $u \in Null(\bar{Q})$*;*

(vi) $\|\bar{x}\| = \bar{r}, a^T\bar{x} + b = 0$, b *is unperturbed and* $det\bar{Q} \neq 0$.

Proof. (i) Since $\|\bar{x}\| < \bar{r}$ and $a^T\bar{x} + \bar{b} < 0$, we have $D^*\mathcal{N}(\bar{\omega})(-y^*) = \{(0_{\mathbb{R}^n}, 0_{\mathbb{R}}, 0_{\mathbb{R}})\}$. Hence (4.39) has a unique solution $(r^*, b^*, y^*) = 0$ and $S(\cdot)$ is locally Lipschitz-like around $(\bar{Q}, \bar{q}, \bar{r}, \bar{b}, \bar{x})$.

(ii) By the assumption that $\|\bar{x}\| = \bar{r}, a^T\bar{x} + \bar{b} < 0$ and $\bar{Q}\bar{x} + \bar{q} = \theta\bar{x}, \theta > 0$, one gets $D^*\mathcal{N}(\bar{p})(v^*) = \Omega_1(\bar{\omega})(-y^*)$ if $\langle -y^*, \bar{x}\rangle = 0$. Then, (4.39) yields

$$\begin{cases} \bar{Q}y^* = 0, \\ 0 = -\frac{r^*}{\bar{r}}\bar{x} - \theta y^*, \\ b^* = 0, \\ \langle y^*, \bar{x}\rangle = 0. \end{cases} \tag{4.41}$$

Combining $-\frac{r^*}{\bar{r}}\bar{x} - \theta y^* = 0$ with $\langle y^*, \bar{x}\rangle = 0$ we have $(y^*, r^*) = 0$. Hence (4.39) has only one solution $(r^*, b^*, y^*) = 0$. This leads to the desired conclusion.

(iii) By the assumption that $\|\bar{x}\| = \bar{r}, a^T\bar{x} + \bar{b} < 0$, and $\bar{Q}\bar{x} + \bar{q} = 0$, we obtain

$$D^*\mathcal{N}(\bar{\omega})(-y^*) = \begin{cases} \{(0_{\mathbb{R}^n}, 0_{\mathbb{R}}, 0_{\mathbb{R}})\} & \text{if } \langle -y^*, \bar{x}\rangle < 0, \\ \Omega_2(\bar{\omega})(-y^*) & \text{if } \langle -y^*, \bar{x}\rangle > 0, \\ \Omega_2'(\bar{\omega})(-y^*) & \text{if } \langle -y^*, \bar{x}\rangle = 0; \end{cases} .$$

Then, (4.39) follows that

$$\begin{cases} \bar{Q}y^* = 0, \\ 0 = -\frac{r^*}{\bar{r}}\bar{x}, \\ b^* = 0, \\ r^* \le 0, \\ \langle y^*, \bar{x}\rangle > 0; \end{cases} \tag{4.42}$$

and

$$\begin{cases} \bar{Q}y^* = 0, \\ 0 = -\frac{r^*}{\bar{r}}\bar{x}, \\ b^* = 0, \\ \langle y^*, \bar{x}\rangle = 0. \end{cases} \tag{4.43}$$

Since $\langle \bar{x}, u\rangle = 0$ for every $u \in Null(\bar{Q})$, (4.42) gives that $\bar{Q}y^* = 0$ and hence $\langle y^*, \bar{x}\rangle = 0$. It follows that (4.42) has no solution. Combining

$\bar{Q}y^* = 0$ and $\langle y^*, \bar{x}\rangle = 0$ with the assumption $rank(\bar{Q}; \bar{x}) = n$, it implies $y^* = 0$. Hence (4.43) has unique solution $(r^*, b^*, y^*) = 0$. Consequently, in this case, (4.39) has only one trivial solution and the conclusion follows.

(iv) Since $\|\bar{x}\| < \bar{r}, a^T\bar{x} + \bar{b} = 0$ and $\bar{Q}\bar{x} + \bar{q} = \gamma a, \gamma > 0$, we have $D^*\mathcal{N}(\bar{p})(v^*) = \Omega_3(\bar{\omega})(-y^*)$ if $\langle -y^*, a\rangle = 0$. Then, (4.39) gives

$$\begin{cases} \bar{Q}y^* = 0, \\ 0 = b^* a, \\ r^* = 0, \\ \langle y^*, a\rangle = 0. \end{cases}$$

From assumption $rank(\bar{Q}; a) = n$, we get $y^* = 0$. Hence (4.39) has a unique solution $(r^*, b^*, y^*) = 0$ and $S(\cdot)$ is locally Lipschitz-like around $(\bar{Q}, \bar{q}, \bar{r}, \bar{b}, \bar{x})$.

(v) Since $\|\bar{x}\| < \bar{r}, a^T\bar{x} + \bar{b} = 0$, and $\bar{Q}\bar{x} + \bar{q} = 0$, we obtain

$$D^*\mathcal{N}(\bar{\omega})(v^*) = \begin{cases} \{(0_{\mathbb{R}^n}, 0_{\mathbb{R}}, 0_{\mathbb{R}})\} & \text{if } \langle v^*, a\rangle < 0, \\ \Omega_4(\bar{\omega})(v^*) & \text{if } \langle v^*, a\rangle > 0, \\ \Omega_3(\bar{\omega})(v^*) & \text{if } \langle v^*, a\rangle = 0. \end{cases}$$

Then, (4.39) yields

$$\begin{cases} \bar{Q}y^* = 0, \\ 0 = b^* a, \\ r^* = 0, \\ b^* \geq 0, \\ \langle y^*, a\rangle > 0; \end{cases} \tag{4.44}$$

and

$$\begin{cases} \bar{Q}y^* = 0, \\ 0 = b^* a, \\ r^* = 0, \\ \langle y^*, a \rangle = 0. \end{cases} \tag{4.45}$$

By the assumption that $\langle \bar{x}, u \rangle = 0$ for every $u \in Null(\bar{Q})$, (4.44) follows $\bar{Q}y^* = 0$. Hence $\langle y^*, a \rangle = 0$. This gives that (4.42) has no solution. Since $rank(\bar{Q}; a) = n$, (4.45) has a unique solution $(r^*, b^*, y^*) = 0$. Hence in this case, (4.39) has only one solution $(r^*, b^*, y^*) = 0$ and the desired conclusion follows.

(vi) From the assumption that $\|\bar{x}\| = \bar{r}$ and $a^T \bar{x} + b = 0$ it follows that $D^*\mathcal{N}(\bar{\omega})(v^*)$ is computed and estimated as in parts (vi)–(ix) of Theorem 4.7.

Since $det\bar{Q} \neq 0$, we now show that (4.40) has unique solution $(r^*, b^*) = 0$. Indeed, from the assumption that b is unperturbed it implies $b^* = 0$. Substituting $b^* = 0$ and $v^* = -y^* = 0$ into the formulas in parts (f)–(i) of Theorem 4.7 yields $r^* = 0$.

Consequently, in this case, (4.40) has only one trivial solution, and $S(\cdot)$ is locally Lipschitz-like around $(\bar{Q}, \bar{q}, \bar{r}, \bar{b}, \bar{x})$. The theorem is proved. $\square$

4.4. Conclusions

In this chapter, we have presented some conditions for the continuity of the optimal value function (Theorem 4.3); the necessary condition for the lower semicontinuity of the stationary solution map (Theorem 4.1); some sufficient conditions for the lower semicontinuity of the stationary solution map (Theorem 4.2); the necessary and sufficient conditions for the lower semicontinuity of the stationary solution map (Theorems 4.4 and 4.5). The Fréchet and Mordukhovich coderivatives of the normal cone mapping related to the parametric ETRS have been computed and estimated (Theorems 4.6 and 4.7). We have used the obtained results and Mordukhovich criterion for the local Lipschitz-like property of multifunctions to estimate the Mordukhovich coderivative of $\widetilde{S}(\cdot)$. Some sufficient conditions for the local Lipschitz-like property of the stationary solution map of parametric ETRS with respect to the linear perturbations have been proposed (Theorems 4.8 and 4.9).

General Conclusions

Our main results for the parametric quadratic programming problems with non-convex objective function include:

1. A Frank-Wolfe type theorem and an Eaves type theorem for solution existence;
2. Conditions for upper and lower semicontinuities of the global and local optimal solution map;
3. Some stability results for the stationary solution set;
4. Conditions for the continuity, Lipschitz property and directional differentiability of the optimal value function;
5. Upper estimations for the Mordukhovich coderivative and conditions for the local Lipschitz-like property of the stationary solution map in parametric extended trust region subproblems.

List of Author's Papers

1. Nghi, T.V., Tam, N.N.: Continuity and directional differentiability of the optimal value function in parametric quadratically constrained nonconvex quadratic programs, *Acta Math. Vietnam.*, 2017, 42(2), 311–336 (**SCOPUS**)

2. Tam, N.N., Nghi, T.V.: On the solution existence and stability of quadratically constrained nonconvex quadratic programs, *Optim. Lett.*, 2017, 1–19, DOI: 10.1007/s11590-017-1163-4 (**SCIE**)

3. Nghi, T.V.: Coderivatives related to parametric extended trust region subproblem and their applications, *Taiwanese J. Math.*, 2017, 1–27, DOI:10.11650/tjm/170907 (**SCI**)

4. Nghi, T.V.: On stability of solutions to parametric generalized affine variational inequalities, *Optimization*, 2017, 1–17, DOI:10.1080/02331934.2017.1394297 (**SCIE**)

5. Nghi, T.V., Tam, N.N.: Stability of the Karush-Kuhn-Tucker point set in parametric extended trust region subproblems (submitted to Acta Math. Vietnam.)

6. Nghi, T.V., Tam, N.N.: Stability for parametric extended trust region subproblems (submitted to Pac. J. Optim.)

Bibliography

[1] Anh, L. Q., Khanh, P. Q.: Continuity of solution maps of parametric quasiequilibrium problems. J. Global Optim. 46, 247–259 (2010)

[2] Anh, L. Q., Khanh, P. Q., Tam, T. N.: On Hlder continuity of approximate solutions to parametric equilibrium problems. Nonlinear Anal. 75, 2293–2303 (2012)

[3] Anh, L. Q., Duy, T. Q.; Khanh, P. Q.: Continuity properties of solution maps of parametric lexicographic equilibrium problems. Positivity 20, 61–80 (2016)

[4] Aubin, J.P., Frankowska, H.: Set-Valued Analysis. Birkhäuser. Berlin (1990)

[5] Auslender, A., Cominetti, R.: First and second order sensitivity analysis of nonlinear programs under directional constraint qualification conditions. Optimization 21, 351–363 (1990)

[6] Auslender, A., and Coutat, P.: Sensitivity analysis for generalized linear-quadratic problems. J. Optim. Theory Appl. 88, 541–559 (1996)

[7] Baccari, A.: On the classical necessary second-order optimality conditions. J. Optim. Theory Appl. 123, 213–221 (2004)

[8] Baccari, A., Trad, A.: On the classical necessary second-order optimality conditions in the presence of equality and inequality constraints. SIAM J. Optim. 15, 394–408 (2004)

[9] Ban, L., Mordukhovich, B.S., Song, W.: Lipschitzian stability of parametric variational inequalities over generalized polyhedra in Banach spaces. Nonlinear Anal. 74, 441–461 (2011)

[10] Bank, B., Guddat, J., Klatte, D., Kummer, B., Tammer, K.: Nonlinear parametric optimization. Akademie-Verlag, Berlin (1982)

[11] Beck, A., Eldar, Y.C.: Strong duality in nonconvex quadratic optimization with two quadratic constraints. SIAM J. Optim. 17, 844-860 (2006)

[12] Belousov, E.G.: Introduction to Convex Analysis and Integer Programming. Moscow University Publ., Moscow (1977) (in Russian)

[13] Belousov, E.G., Klatte, D.: A Frank-Wolfe type theorem for convex polynomial programs. Comp. Optim. Appl. 22, 37–48 (2002)

[14] Bertsekas, D.P., Tseng, P.: Set intersection theorems and existence of optimal solutions. Math. Program. 110, 287–314 (2007)

[15] Best, M. J., Chakravarti, N.: Stability of linearly constrained convex quadratic programs. J. Optim. Theory Appl. 64, 43–53 (1990)

[16] Best, M.J., Ding, B.: On the continuity of minimum in parametric quadratic programs. J. Optim. Theory Appl. 86, 245–250 (1995)

[17] Bomze, I.: Copositivity for second-order optimality conditions in general smooth optimization problems. Optimization 65, 779–795 (2016)

[18] Bonnans, J.F., Shapiro, A.: Perturbation Analysis of Optimization Problems. Springer-Verlag, New York (2000)

[19] Borwein, J.M.: Necessary and sufficient conditions for quadratic minimality. Numer. Funct. Anal. Optim. 5, 127–140 (1982)

[20] Brezis, H.: Analyse fonctionnelle, 2^e tirage, Masson, Paris (1987)

[21] Burer, S., Anstreicher, K.M.: Second-order cone constraints for extended trust region problems. SIAM J. Optim. 23, no 1, 432–451 (2013)

[22] Burer, S., Yang, B.: The trust region subproblem with non-intersecting linear constraints. Math. Program. 149, no. 1-2, Ser. A, 253–264 (2015)

[23] Clarke, F. H.: Optimization and nonsmooth analysis. John Wiley & Sons, New York (1983)

[24] Conn, A.R., Gould, N.I.M., Toint, P.L.: Trust-Region Methods. MPS-SIAM Series on Optimization, Philadelphia (2000)

[25] Contesse, L.B.: Une caractérisation complète des minima locaux [A complete characterization of local minima]. Numer. Math. 34, 315–332 (1980)

[26] Cottle, R.W., Pang, J.S., Stone, R.E.: The Linear Complementarity Problem. Academic Press, New York (1992)

[27] Daniel, J.W.: Stability of the solution of deflnite quadratic programs. Math. Program. 5, 41–53 (1973)

[28] Dempe, S., Mehlitz, P.: Lipschitz continuity of the optimal value function in parametric optimization. J. Glob. Optim. 61, no. 2, 363–377 (2015)

[29] Dontchev, A.L., Rockafellar, R.T.: Characterization of strong regularity for variational inequalities over polyhedral convex sets. SIAM J. Optim. 6, 1087–1105 (1996)

[30] Dontchev, A.L., Rockafellar, R.T.: Implicit Functions and Solution Mappings: A View from Variational Analysis. Springer, New York (2009)

[31] Eaves, B.C.: On quadratic programming. Manag. Sci. 17, 698–711 (1971)

[32] Facchinei, F., Pang, J.S.: Finite-Dimensional Variational Inequalities and Complementary Problems, Vol. I. Springer, New York (2003)

[33] Floudas, C.A., Visweswaran, V.: Quadratic Optimization, in: R. Horst and P. M. Pardalos (eds.) Handbook of Global Optimization. Springer 217–264 (1995)

[34] Frank, M., Wolfe, P.: An algorithm for quadratic programming. Naval Res. Logist. Quar. 3, 95–110 (1956)

[35] Gauvin, J.: A necessary and sufficient regularity condition to have bounded multipliers in nonconvex programming, Math. Program. 12, 136–138 (1977)

[36] Gauvin, J., Dubeau, F.: Differential properties of the marginal function in mathematical programming. Math. Program. Study 19, 101–119 (1982)

[37] Gauvin, J., Janin, R.: Directional derivative of the optimal value function in parametric optimization. Ann. Oper. Res. 27, 237–252 (1990)

[38] Giorgi, G., Guerraggio, A., Thierfelder, J.: Mathematics of Optimization: Smooth and Nonsmooth Case. Elsevier Science (2004)

[39] Gollan, B.: On the marginal function in nonlinear programming. Math. Oper. Res. 9, 208–221 (1984)

[40] Gowda, M.S., Pang, J.S.: On the boundedness and stability of solutions to the affine variational inequality problem. SIAM J. Control Optim. 32, 421–441 (1994)

[41] Gowda, M.S., Seidman, T.S.: Generalized linear complementarity problems. Math. Program. 46, 329-340 (1990)

[42] Gfrerer, H., Outrata, J.V.: On computation of limiting coderivatives of the normal-cone mapping to inequality systems and their applications. Optimization 65, 671–700 (2016)

[43] Guddat, J.: Stability in convex quadratic programming. Math. Operations forsch. Stat. 7, 223–245 (1976)

[44] Han, Y., Chen, Z., Zhang, F.: Lipschitz continuity of the optimal value function and the stationary solution set in indefinite quadratic programs. Appl. Math. J. Chinese Univ. 30, 102–110 (2015)

[45] Henrion, R., Mordukhovich, B.S., Nam, M.N.: Second-order analysis of polyhedral systems in finite and infinite dimensions with applications to robust stability of variational inequalities. SIAM J. Optim. 20, 2199–2227 (2010)

[46] Henrion, R., Outrata, J.: On calculating the normal cone to a finite union of convex polyhedral. Optimization 57, 57–78 (2008)

[47] Huy, N.Q., Yao, J.C.: Exact formulas for coderivatives of normal cone mappings to perturbed polyhedral convex sets. J. Optim. Theory Appl. 157, 25–43 (2013)

[48] Jeyakumar, V., Li, G.Y.: Trust-region problems with linear inequality constraints: exact SDP relaxation, global optimality and robust optimization. Math. Program. 147, no. 1-2, Ser. A, 171–206 (2014)

[49] Kim, D.S., Tam, N.N., Yen, N.D.: Solution existence and stability of quadratic of quadratically constrained convex quadratic programs. Optim. Lett. 6, 363–373 (2012)

[50] Kien, B. T.: Lower semicontinuity of the solution map to a parametric generalized variational inequality in reflexive Banach spaces. Set-Valued Anal. 16, 1089–1105 (2008)

[51] Kinderlehrer, D., Stampacchia, G.: An Introduction to Variational Inequalities and Their Applications. Academic Press, New York (1980)

[52] Klatte, D.: On the Lipschitz behavior of optimal solutions in parametric problems of quadratic optimization and linear complementarity. Optimization 16, 819–831 (1985)

[53] Klatte, D., Tammer, K.: Strong stability of stationary solutions and Karush-Kuhn-Tucker points in nonlinear optimization. Annal. Oper. Research 27, 285–308 (1990)

[54] Kojima, M.: Strongly stable stationary solutions in nonlinear programs, in: S.M. Robinson, ed., Analysis and Computation of Fixed Points (Academic Press, New York, 1980) 93–138

[55] Kuhn, H.W., Tucker, A.W.: Nonlinear Programming, in: J. Neyman (Ed.), Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability. Berkeley, CA: University of California Press, 481–492 (1951)

[56] Lee, G.M., Tam, N.N., Yen, N.D.: Quadratic programming and affine variational inequalities: a qualitative study, Series: Nonconvex Optimization and Its Application, vol. 78. Springer-Verlag, New York (2005)

[57] Lee, G.M., Tam, N.N., Yen, N.D.: On the optimal value function of a linearly perturbed quadratic program. J. Glob. Optim. 32, 119–134 (2005)

[58] Lee, G.M., Tam, N.N., Yen, N.D.: Continuity of the solution map in quadratic programs under linear perturbations. J. Optim. Theory Appl. 129, 415–423 (2006)

[59] Lee, G.M., Tam, N.N., Yen, N.D.: Lower semicontinuity of the stationary solution set in quadratic programs under linear perturbations. Acta Math. Vietnam. 34, 411–422 (2006)

[60] Lee, G.M., Tam, N.N., Yen, N.D.: Continuity of the solution map in parametric affine variational inequalities. Set-Valued Anal. 15, 105–123 (2007)

[61] Lee, G.M., Tam, N. N., Yen, N.D.: Stability of linear-quadratic minimization over Euclidean ball. SIAM J. Optim. 22, 936–952 (2012)

[62] Lee, G.M., Yen, N.D.: Fréchet and normal coderivatives of implicit multifunctions. Appl. Anal. 90, 1011–1027 (2011)

[63] Lee, G.M., Yen, N.D.: Coderivatives of a Karush-Kuhn-Tucker point set map and applications. Nonlinear Anal. 95, 191–201 (2014)

[64] Luderer, B., Minchenko, L.I., Satsura,T.: Multivalued Analysis and Nonlinear Programming Problems with Perturbations. Kluwer, Dordrecht (2002)

[65] Luo, Z.Q., Pang, J.S., Ralph, D.: Mathematical Programs with Equilibrium Constraints. Cambridge University Press, Cambridge (1996)

[66] Luo, Z.Q., Zhang, S.: On extensions of the Frank-Wolfe theorems. Comput. Optim. Appl. 13, 87–110 (1999)

[67] Lucidi, S., Palagi, L., Roma, M.: On some properties of quadratic programs with a convex quadratic constraint. SIAM J. Optim. 8, 105–122 (1998)

[68] Majthay, A.: Optimality conditions for quadratic programming. Math. Program. 1, 359–365 (1971)

[69] Mangasarian, O. L.: Locally unique solutions of quadratic programs, linear and nonlinear complementarity problems. Math. Program. 19, 200–212 (1980)

[70] Martinez, J. M.: Local minimizers of quadratic functions on Euclidean balls and spheres. SIAM J. Optim. 4, 159–176 (1994)

[71] Minchenko, L.I., Sakochik, P.P.: Holder behavior of optimal solutions and directional differentiability of marginal functions in non linear programming. J. Optim. Theory Appl. 90, 555–580 (1996)

[72] Minchenko, L.I., Tarakanov, A.: On second-order directional derivatives of optimal value functions. Optimization 64, 389–407 (2015)

[73] Mordukhovich, B.S.: Variational Analysis and Generalized Differentiation. Vol. I: Basic Theory. Springer-Verlag, Berlin (2006)

[74] Nam, N.M.: Coderivatives of normal cone mappings and Lipschitzian stability of parametric variational inequalities. Nonlinear Anal. 73, 2271–2282 (2010)

[75] Nghi, T.V., Tam, N.N.: Continuity and directional differentiability of the optimal value function in parametric quadratically constrained nonconvex quadratic programs. Acta Math. Vietnam. 42(2), 311–336 (2017)

[76] Nghi, T.V.: Coderivatives related to parametric extended trust region subproblem and their applications, Taiwanese J. Math., 2017, 1–27, DOI:10.11650/tjm/170907

[77] Nghi, T.V.: On stability of solutions to parametric generalized affine variational inequalities, Optimization, 2017, 1–17, DOI:10.1080/02331934.2017.1394297

[78] Nghi, T.V., Tam, N.N.: Stability of the Karush-Kuhn-Tucker point set in parametric extended trust region subproblems (submitted)

[79] Nghi, T.V, Tam, N.N.: Stability for parametric extended trust region subproblems (submitted)

[80] Nhan, N.H.: On the Stability of Linear Complementarity Problems and Quadratic Programming Problems. Master thesis, University of Hue, Hue (1995)

[81] Orden, J.: Stationary points of quadratic functions under linear constraints. Comput. J. 7, 238–242 (1964)

[82] Pardalos, P., Romeijn, H.: Handbook in Global Optimization 2, Kluwer Academic Publishers (2002)

[83] Qui, N.T.: Linearly perturbed polyhedral normal cone mappings and applications. Nonlinear Anal. 74, 1676–1689 (2011)

[84] Qui, N.T.: New results on linearly perturbed polyhedral normal cone mappings. J. Math. Anal. Appl. 381, 352–364 (2011)

[85] Qui, N.T., Yen, N.D.: A class of linear generalized equations, SIAM J. Optim. 24, 210–231 (2014)

[86] Qui, N.T.: Generalized differentiation of a class of normal cone operators. J. Optim. Theory Appl. 161, 398–429 (2014)

[87] Robinson, S.M.: Stability theory for systems of inequalities. Part I: Linear systems. SIAM J. Numer. Anal. 12, 754–769 (1975)

[88] Robinson, S.M.: Generalized equations and their solutions, Part I: Basic theory. Math. Program. Study 10, 128–141 (1979)

[89] Robinson, S.M.: Strongly regular generalized equations. Math. Oper. Res. 5, 43–62 (1980)

[90] Rockafellar, R.T.: Convex Analysis. Princeton University Press, Princeton, New Jersey (1970)

[91] Rockafellar, R.T., Wets, R.J.B.: Variational Analysis. Springer, Berlin, Heidelberg (1998)

[92] Salahi, M., Fallahi, S.: Trust region subproblem with an additional linear inequality constraint. Optim. Lett. 10, 821–832 (2016)

[93] Salahi, M., Taati, A., Wolkowicz, H.: Local Nonglobal Minima for Solving Large Scale Extended Trust Region Subproblems. Comput. Optim. Appl. 66, 223–244 (2017)

[94] Seeger, A.: Second order directional derivatives in parametric optimization problems. Math. Oper. Res. 13, 124–139 (1988)

[95] Sun, W.-Y, Yuan,Y.-X.: Yuan, Optimization Theory and Methods: Nonlinear Programming. Springer, New York (2006)

[96] Tam, N.N.: On continuity properties of the solution map in quadratic programming. Acta Math. Vietnam. 24, no. 1, 47–61 (1999)

[97] Tam, N.N.: Stability for quadratic programming. Doctor dissertation in mathematics, VAST, Hanoi (1999)

[98] Tam, N.N.: Directional differentiability of the optimal value function in indefinite quadratic programming. Acta Math. Vietnam. 26, 377–394 (2001)

[99] Tam, N.N.: Sufficient conditions for the stability of the Karush-Kuhn-Tucker point set in quadratic programming. Optimization 50, 45–60 (2001)

[100] Tam, N.N.: Continuity of the optimal value function in indefinite quadratic programming. J. Glob. Optim. 23, 43–61 (2002)

[101] Tam, N.N.: Some stability results for the semi-affine variational inequality problem. Acta Math. Vietnam. 29, 271-280 (2004)

[102] Tam, N.N., Nghi, T.V.: On the solution existence and stability of quadratically constrained nonconvex quadratic programs, Optim. Lett., 2017, 1–19, DOI: 10.1007/s11590-017-1163-4

[103] Tam, N.N., Yen, N.D.: Continuity properties of the Karush-Kuhn-Tucker point set in quadratic programming problems. Math. Program. 85, 193–206 (1999)

[104] Tam, N.N., Yen, N.D.: Stability of the Karush-Kuhn-Tucker point set in a general quadratic programming problem, Vietnam J. Math. 28, 67–79 (2000)

[105] Terlaky, T.: On l_p programming. Eur. J. Oper. Res. 22, 70–100 (1985)

[106] Tuy, H., Tuan, H.D.: Generalized S-lemma and strong duality in nonconvex quadratic programming. J. Glob. Optim. 56(3), 1045–1072 (2013)

[107] Ye, Y.X., Zhang, S.Z.: New results on quadratic minimization. SIAM J. Optim. 14, 245–267 (2003)

[108] Yen, N.D: Parametric optimization problems and parametric variational inequalities. Vietnam J. Math. 37, 191–223 (2009)

[109] Zeidler, E.: Nonlinear Functional Analysis and its Applications. I: Fixed Point Theorems. Springer, New York (1986)