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EXISTENCE AND STABILITY FOR QUADRATIC PROGRAMMING PROBLEMS WITH NON-CONVEX OBJECTIVE FUNCTION

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SUMMARY OF DOCTORAL DISSERTATION IN MATHEMATICS

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Introduction

Quadratic programming (QP) problems constitute a special class of *nonlinear programming* (NLP) problems. Numerous problems in real world applications, including problems in planning and scheduling, economies of scale, engineering design, and control are naturally expressed as QP problems. One also uses QP problems in order to approximate problems.

Many important research results for *linearly constrained quadratic programming* (LCQP) problems can be found in Lee et al. (2005) and the references cited therein. Since the finite dimensional LCQP problems have been rather comprehensively investigated, several authors are now interested in studying *quadratically constrained quadratic programming* (QCQP) problems.

The solution existence of QP problems is one of the most important issues. Frank and Wolfe (1956) extended the fundamental existence of linear programming by proving that an arbitrary quadratic function f attains its minimum over a nonempty convex polyhedral set C provided f is bounded from below over C (called *Frank-Wolfe Theorem*). From then to now, there have been some other proofs for this theorem and its extended versions (Belousov (1977), Terlaky (1985)). In 1999, Luo and Zhang proved that a QP problem has a solution if its objective function is bounded below over a nonempty constraint set defined by a convex quadratic function and linear constraint functions. They also showed that there exists a nonconvex QP problem in $\mathbb{R}^4$ with two convex quadratic constraints whose objective function is bounded from below over a nonempty constraint set, which has no solutions. Belousov and Klatte (2002) observed that the effect of nonconvexity of the objective function can be seen in $\mathbb{R}^3$. Bertsekas and Tseng (2007) proved the solution existence of a QP problem when all the asymptotic directions of constraint set are retractive local horizon directions and the objective function is bounded below constraint set. By using the concept of recession cone in convex analysis, Lee et al. (2012) established an Eaves type Theorem for convex QCQP problems. Up to now, many researchers have been studying sufficient conditions for the solution existence of a nonconvex QP problem whose constraint set is defined by finitely many quadratic inequalities.

Stability for parametric QCQP problems plays an important role because they can be used for analyzing algorithms for solving this problem. For convex QP problems, Best et al. (1990, 1995) obtained some results on the continuity and differentiability of the optimal value function; continuity and/or differentiability properties of the global solution map have been discussed (see, for example, Auslender and Coutat (1996), Best and Chakravarti (1990), Cottle et al. (1992), Daniel (1973), Guddat (1976), Robinson (1979). For nonconvex LCQP problems, the continuity for the global solution map, stationary solution map and the optimal value function have been investigated in details by Lee et al. (2005) and the references therein. For TRSs, Lee et al. (2012) investigated the case where the linear part or the quadratic form of the objective function is perturbed and obtain necessary and sufficient conditions for the upper/lower semicontinuity of the stationary solution map and the global solution map, explicit formulas for computing the directional derivative and the Fréchet derivative of the optimal value function. Lee and Yen (2011) estimated the Mordukhovich coderivative and conditions for the local Lipschitz-like property of the stationary solution map in parametric TRS. Since QP is a class of nonlinear optimization problems, the results in nonlinear optimization can be applied to convex and nonconvex QP problems. Differential properties of the marginal function and of the global solution map in mathematical programming were investigated by Gauvin and Dubeau (1982). Continuity and Lipschitzian properties of the optimal value function have been studied by Bank et al. (1982), Rockafellar and Wets (1998). Auslender and Cominetti (1990) considered first and second-order sensitivity analysis of NLP under directional constraint qualification conditions. Minchenko and Tarakanov (2015) discussed directional derivatives of the optimal value functions under the assumption of the calmness of global solution mapping. Lipschitzian continuity of the optimal value function was presented by Dempe and Mehlitz (2015). Some similar topics related to Lipschitzian stability have been investigated in Gauvin and Janin (1990), Luderer et al. (2002), Minchenko and Sakochik (1996), Seeger (1988) and the references given there. A survey of recent results on stability of NLP problems was given by Bonnans and Shapiro (2000). In which, many interesting results can be applied for QP. However, the special structure of QP problems allows one to have deeper and more comprehensive results on stability in QCQP.

This dissertation gives new results on *the existence and stability for quadratic programming problems with non-convex objective function.* By using the special structure of quadratic forms, the recession cone and some advanced tools of variational analysis, we propose conditions for the solution existence and investigate in details the stability for QCQP problems. The specific techniques and theoretical results for LCQP and TRS cannot be directly applied, and a more general approach is used. Among our proposed assumptions, there are some weaker than ones used in the cited works (applied for QP). We also generalize some stability results from the case of polyhedral convex constraint set to the case of constraint set defined by finitely many convex quadratic functions.

The dissertation has four chapters and a list of references.

Chapter 1 presents sufficient conditions for the solution existence of QCQP problems through a Frank-Wolfe type Theorem and an Eaves type Theorem.

Chapter 2 investigates the continuity of the global, local and stationary solution maps of parametric QCQP problems by using the obtained results on solution existence.

Chapter 3 characterizes the continuity, Lipschitzian continuity and directional differentiability of the optimal value function under weaker assumptions in comparison with results which are implied from general theory.

Chapter 4 describes the special stability properties of parametric extend trust region subproblems (ETRS). We estimate the Mordukhovich coderivative of the stationary solution map and investigate Lipschitzian stability for parametric ETRS.

The dissertation is written on the basis of the paper [1] in Acta Math. Vietnam., the paper [2] in Optim. Lett., and preprints [3–6], which have been submitted.

The results of this dissertation were presented at International Workshop on New Trends in Optimization and Variational Analysis for Applications (Quynhon, December 7–10, 2016); The 14^{th} Workshop on Optimization and Scientific Computing (Bavi, April 21–23, 2016); The 5^{th} National Workshop of young researchers in teacher training university (Vinhphuc, May 23–24, 2015); Scientific Conference at Hanoi Pedagogical University 2 (HPU2) (Vinhphuc, November 14, 2015); at the seminar of Department of Mathematics, HPU2 and at the seminar of the Department of Numerical Analysis and Scientific Computing, Institute of Mathematics, Vietnam Academy of Science and Technology.

Chapter 1

Existence of solutions

The aim of this chapter is to investigate existence of solutions of QCQP problems. The presentation given below comes from the results in [2].

1.1. Problem statement

Let $\mathbb{R}^n$ be n-dimensional Euclidean space equipped with the standard scalar product and the Euclidean norm, $\mathbb{R}^{n\times n}_S$ be the space of real symmetric $(n \times n)$–matrices equipped with the matrix norm induced by the vector norm in $\mathbb{R}^n$ and $\mathbb{R}^{n\times n}_{S^+}$ be the set of positive semidefinite real symmetric $(n \times n)$–matrices. Let

$$\mathbb{P} := \mathbb{R}^{n\times n}_S \times \mathbb{R}^n \times \underbrace{(\mathbb{R}^{n\times n}_{S^+} \times \mathbb{R}^n \times \mathbb{R}) \times \ldots \times (\mathbb{R}^{n\times n}_{S^+} \times \mathbb{R}^n \times \mathbb{R})}_{m\ times} \subset \mathbb{R}^s$$

with $s = (m + 1)(n^2 + n + 1) - 1$. The scalar product of vectors x, y and the Euclidean norm of a vector x in a finite-dimensional Euclidean space are denoted, respectively, by $x^T y$ (or $\langle x, y\rangle$) and $\|x\|$, where the superscript ^{T} denotes the transposition.

Let us consider the following nonconvex QCQP problem

$$\begin{array}{l} \min f(x,p) = \frac{1}{2}x^T Qx + q^T x \\ s.t.\ x \in \mathbb{R}^n : g_i(x,p) = \frac{1}{2}x^T Q_i x + q_i^T x + c_i \leq 0, i = 1,\ldots,m, \end{array} \qquad (QP(p))$$

depending on the parameter $p = (Q, q, Q_1, q_1, c_1, \ldots, Q_m, q_m, c_m) \in \mathbb{P}$.

The *feasible region*, the *local solution set* and the *global solution set* of $(QP(p))$ will be denoted by $\mathcal{F}(p)$, $L(p)$, and $G(p)$, respectively.

The *recession cone* of $\mathcal{F}(p) \neq \emptyset$ is defined by

$$0^+ \mathcal{F}(p) = \{v \in \mathbb{R}^n : x + tv \in \mathcal{F}(p)\ \ \forall x \in \mathcal{F}(p)\ \ \forall t \geq 0\}.$$

According to Kim et al. (2012), we obtain that

$$0^+ \mathcal{F}(p) = \{v \in \mathbb{R}^n : Q_i v = 0, q_i^T v \leq 0,\ \forall i = 1,\ldots,m\}.$$

The function

$$\varphi : \mathbb{P} \longrightarrow \mathbb{R} \cup \{\pm\infty\}$$

defined by

$$\varphi(p) = \begin{cases} \inf\{f(x,p) : x \in \mathcal{F}(p)\} & \text{if } \mathcal{F}(p) \neq \emptyset; \\ +\infty & \text{if } \mathcal{F}(p) = \emptyset, \end{cases}$$

is called *the optimal value function* of the parametric problem $(QP(p))$.

1.2. A Frank-Wolfe type theorem

Fix $p \in \mathbb{P}$ and let

$$I = \{1, \ldots, m\},\ I_0 = \{i \in I :\ Q_i = 0\},\ I_1 = \{i \in I :\ Q_i \neq 0\} = I \setminus I_0.$$

The following result is a generalization of Frank-Wolfe Theorem.

Theorem 1.1. *Consider the problem $(QP(p))$. Assume that $\mathcal{F}(p)$ is nonempty, $f(x,p)$ is bounded from below over $\mathcal{F}(p)$ and one of the following conditions is satisfied:*
(A_1) *The set I_1 consists of at most one element;*
(A_2) *If $v \in 0^+\mathcal{F}(p)$ such that $v^TQv = 0$ then $q_i^Tv = 0$ for all $i \in I_1$.*
Then, $(QP(p))$ has a solution .

We obtain some important consequences of Theorem 1.1.

Corollary 1.1. (Frank-Wolfe Theorem) *Consider the LCQP problem (that is, $(QP(p))$ with $Q_i = 0$ for all $i = 1, ..., m$). Assume that $f(x,p)$ is bounded from below over nonempty $\mathcal{F}(p)$. Then, the problem* (LCQP) *has a solution .*

Corollary 1.2. *Assume that the function $f(x,p) = \frac{1}{2}x^TQx + q^Tx$ is bounded from below over $\mathbb{R}^n$. Then, there exists an $x^* \in \mathbb{R}^n$ such that $f(x^*,p) \leq f(x,p)$ for all $x \in \mathbb{R}^n$.*

Corollary 1.3. *Consider $(QP(p))$. If $\mathcal{F}(p)$ is nonempty and $v^TQv > 0$ for every nonzero vector $v \in 0^+\mathcal{F}(p)$ then $G(p)$ is a nonempty compact set.*

1.3. An Eaves type theorem

Eaves (1971) presented another fundamental existence theorem (called *Eaves Theorem*) for LCQP problems which gives us a tool for checking the boundedness from below of the object function on constraints set.

Unlike the case of LCQP, Eaves type necessary conditions for solution existence of $(QP(p))$ do not coincide with the sufficient ones. The following result is a generalization of Eaves Theorem.

Theorem 1.2. *Consider $(QP(p))$ and assume that $\mathcal{F}(p)$ is nonempty. The following statements are valid:*

a) If $(QP(p))$ has a solution, then

$$i)\ v^TQv \geq 0 \ \ \forall v \in 0^+\mathcal{F}(p), \tag{1.17}$$

$$ii)\ (Qx+q)^Tv \geq 0 \ \ \forall x \in \mathcal{F}(p) \forall v \in \{u \in 0^+\mathcal{F}(p) : u^TQu = 0\}. \tag{1.18}$$

b) If (1.17), (1.18), and (A_2) hold, then $(QP(p))$ has a solution.

The following example shows that (A_2) can not be dropped from the assumptions in Theorem 1.2.

Example 1.1. *Consider the following problem*

$$\min\{f(x,p) = -2x_2x_3 + 2x_1 : x \in \mathcal{F}(p)\},$$

where $\mathcal{F}(p) = \{(x_1,x_2,x_3) \in \mathbb{R}^3 : x_2^2 - x_1 \leq 0; x_3^2 - x_1 - 1 \leq 0\}$. Both conditions (1.17) and (1.18) are satisfied. Condition (A_2) is not satisfied and this problem has no solution.

To illustrate for Theorem 1.2, we consider the following example.

Example 1.2. *Let us consider the problem $(QP(p))$ with $m = 2$, $n = 3$, and*

$$Q = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad q = \begin{pmatrix} 0 \\ 2 \\ -5 \end{pmatrix}, \qquad Q_1 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$q_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad c_1 = 0, \qquad Q_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad q_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad c_2 = 0.$$

According to Theorem 1.2, this problem has a solution.

Chapter 2

Stability for global, local and stationary solution sets

In this chapter, we characterize continuity of the global, local and stationary solution maps. The material of this chapter is taken from [2,4,6].

2.1. Continuity of the global solution map

Using the obtained results on solution existence in Chapter 1, this section characterizes continuity of the global solution map of QCQP problems. First of all, we present the following assumptions and auxiliary results.

2.1.1. Assumptions and auxiliary results

An important assumption used in our proof is given below.
Assumption $(\mathbf{A}_3)$ *The set* $\mathcal{F}(p) \neq \emptyset$ *and* $v^T Q v > 0$ *for every nonzero vector* $v \in 0^+\mathcal{F}(p)$.

Clearly, (A_3) holds if $\mathcal{F}(p)$ is nonempty and bounded. Thus (A_3) is weaker than the uniform compactness of C near p in Gauvin and Dubeau (1982) applied for $(QP(p))$.

2.1.2. Upper semicontinuity of the global solution map

The upper semicontinuity of the global solution map $G(\cdot)$ is characterized as follows.

Theorem 2.1. *Consider the problem* $(QP(p))$ *and* $\bar{p} \in \mathbb{P}$. *Assume that* (SCQ) *and* (A_3) *hold at* $\bar{p}$. *Then,* $G(\cdot)$ *is upper semicontinuous at* $\bar{p}$.

2.1.3. Lower semicontinuity of the global solution map

The following theorem shows the necessary and sufficient condition for the lower semicontinuity of the global solution map $G(\cdot)$.

Theorem 2.2. *Consider the problem $(QP(p))$ and $\bar{p} \in \mathbb{P}$. The global solution map $G(\cdot)$ is lower semicontinuous at $\bar{p}$ if and only if (SCQ) and (A_3) hold at $\bar{p}$ and $G(\bar{p})$ is a singleton.*

2.2. Continuity of the local solution map

In this section, we propose a necessary and sufficient condition for the lower semicontinuity of the local solution map $L(\cdot)$. The *isolated local solution set* of $(QP(p))$ will be denoted by $IL(p)$. The main result is presented below.

Theorem 2.3. *The local solution map $L(\cdot)$ is lower semicontinuous at $\bar{p}$ if and only if $(QP(\bar{p}))$ satisfies (SCQ) and the set of local solutions coincides with the set of isolated local solutions, i.e., $L(\bar{p}) = IL(\bar{p})$.*

2.3. Stability of stationary solutions

In this section, the upper semicontinuity of the stationary solution map is characterized. A stability result for stationary solution set is also investigated in the connection with parametric extended affine variational inequalities.

2.3.1. Preliminaries

Recall that x is a *stationary solution* of the problem $(QP(p))$ if there exists Lagrange multiplier $\lambda \in \mathbb{R}^m$ satisfying the following *Karush-Kuhn-Tucker (KKT) condition*:

$$\begin{aligned}
&Qx + q + \sum_{i=1}^{m} \lambda_i (Q_i x + q_i) = 0, \\
&\lambda \geq 0,\ g_i(x,p) \leq 0, \\
&\lambda_i g_i(x,p) = 0,\ i = 1, \ldots, m.
\end{aligned}$$

The stationary solution set of $(QP(p))$ is denoted by $S(p)$. It is well-known that under (SCQ),

$$G(p) \subset L(p) \subset S(p) \subset \mathcal{F}(p).$$

2.3.2. Upper semicontinuity of the stationary solution map

Denote

$$Null(Q) := \{x \in \mathbb{R}^n : Qx = 0\}.$$

The following result gives a sufficient condition for the upper semicontinuity of the stationary solution map $G(\cdot)$.

Theorem 2.4. *Consider the problem $(QP(p))$ and $\bar{p} \in \mathbb{P}$. If $(QP(\bar{p}))$ satisfies (SCQ) and*

$$Null(\bar{Q}) \cap 0^+\mathcal{F}(\bar{p}) = \{0\},$$

then $S(\cdot)$ is upper semicontinuous at $\bar{p}$.

Remark 2.1. *According to Theorem 2.4, the assumption (SCQ) is also a sufficient condition for the upper semicontinuity of the stationary solution map $S(\cdot)$ in the case where any component of p is perturbed. But the reverse, in general, is not true.*

The following is an immediate consequence of Theorem 2.4.

Corollary 2.1. *Consider the problem $(QP(p))$ and $\bar{p} \in \mathbb{P}$. If $(QP(\bar{p}))$ satisfies (SCQ) and one of the following conditions is satisfied:*

(i) $\mathcal{F}(\bar{p})$ is bounded;

(ii) $\bar{Q}$ is nonsingular (that is, $\det \bar{Q} \neq 0$),

then $S(\cdot)$ is upper semicontinuous at $\bar{p}$.

2.3.3. A result on stability of stationary solutions

In this section, we presents a result on stability of the stationary solution set. We use the tools relate to *extended affine variational inequality (EAVI)* to prove the main result.

Let $S \subset \mathbb{R}^n$ be a closed convex set and F be a function on S. A *variational inequality (VI)* problem has the following form

$$\textit{Find } x \in S \textit{ such that } \langle F(x), y - x\rangle \geq 0 \ \ \forall y \in S. \tag{$VI(F,S)$}$$

$VI(F,S)$ reduces to the *affine variational inequality (AVI) problem* if S is a polyhedral convex set and $F(x) = Qx + q$ with Q being an $(n \times n)$-matrix and $q \in \mathbb{R}^n$. The stability of the AVI problems has been studied by many authors. Gowda and Pang (1994) obtained several sufficient conditions for the boundedness and stability of solutions to the AVI problem. Robinson (1979) studied the stability of the AVI problems by the nonemptiness and the boundedness of global solution set for the case where Q is a positive semidefinite matrix. Some similar topics have been investigated by Gowda and Seidman (1990). Lee et al. (2007) presented conditions for the upper and the lower semicontinuities of the solution map of AVI problems. Some Lipschitz continuous properties of the solution map of the AVI problem were discussed in Lee et al. (2005).

As $F(x) = Qx + q$ and S is an arbitrary closed convex set, $VI(F,S)$ reduces to the EAVI problem. Tam (2004) presented some stability results for the EAVI problem. A survey on the parametric optimization problems and parametric variational inequalities was given by Yen (2009).

In this section, we concern the EAVI problem as follows

$$\textit{Find } x \in \mathcal{F}(p) \textit{ such that } \langle Qx + q, y - x\rangle \geq 0 \ \ \forall y \in \mathcal{F}(p) \tag{$VI(p)$}$$

depending on the parameter $p \in \mathbb{P}$. The *solution set* of $VI(p)$ will be denoted by $SolVI(p)$. Under the assumption (SCQ), we have

$$S(p) = SolVI(p).$$

The following theorem is the main result in this subsection.

Theorem 2.5. *Consider the problem $(QP(p))$ and $\bar{p} \in \mathbb{P}$. Assume that $(QP(\bar{p}))$ satisfies (SCQ) and the following two conditions are satisfied:*

(a_1) $\{h \in 0^+\mathcal{F}(\bar{p}) : h^T\bar{Q}h = 0\} \subset Null(\bar{Q})$;

(a_2) *If $\mathcal{F}(\bar{p})$ is unbounded then*

$$\limsup_{k\to\infty} \frac{(x^k)^T\bar{Q}x^k}{\|x^k\|^2} \geq 0$$

for every sequence $\{x^k\} \subset \mathcal{F}(\bar{p})$ satisfying $\|x^k\| \to \infty$.

Then, the following four assertions are equivalent:

(b_1) *There exists a number* $\gamma > 0$ *such that* $S(\tilde{p})$ *is nonempty for every* $\tilde{p} \in \mathbb{P}$ *satisfying* $\|\tilde{p} - \bar{p}\| < \gamma$*;*

(b_2) $S(\bar{p})$ *is nonempty and bounded;*

(b_3) $\big\{x \in \mathcal{F}(\bar{p}) : (\bar{Q}x + \bar{q})^T h > 0 \;\; \forall h \in 0^+\mathcal{F}(\bar{p}) \setminus \{0\}\big\} \neq \emptyset$*;*

(b_4) $\bar{q} \in int((0^+\mathcal{F}(\bar{p}))^* - \bar{Q}\mathcal{F}(\bar{p}))$,
where $(0^+\mathcal{F}(\bar{p}))^* = \{y \in \mathbb{R}^n : h^T y \geq 0 \;\; \forall h \in 0^+\mathcal{F}(\bar{p})\}$.

Remark 2.2. *The assumption* (a_2) *is weaker than the assumption* (ii) *which proposed by Tam (2004).*

Chapter 3

Continuity and directional differentiability of the optimal value function

This chapter deals with continuity and directional differentiability of the optimal value function in nonconvex QCQP problems. Among our proposed assumptions, there are some weaker than the assumptions used in the cited works (applied for QCQP). This chapter is written on the basis of the results in [1, 2].

3.1. Continuity of the optimal value function

The following theorem shows the necessary and sufficient condition for continuity of the optimal value function.

Theorem 3.1. *Consider* $(QP(p))$ *and* $\bar{p} \in \mathbb{P}$. *Assume that* f *is bounded from below over* $\mathcal{F}(\bar{p}) \neq \emptyset$. *Then,* φ *is continuous at* $\bar{p}$ *if and only if* (SCQ) *and* (A_3) *are fulfilled at* $\bar{p}$.

Stability and Lipschitzian stability for parametric nonconvex QCQP problem are characterized as follows.

Theorem 3.2. *Consider* $(QP(p))$ *and* $\bar{p} \in \mathbb{P}$. *Assume that* (SCQ) *and* (A_3) *hold at* $\bar{p}$. *Then, the following four statements are equivalent:*

(a) $G(\cdot)$ *is lower semicontinuous at* $\bar{p}$*;*

(b) $G(\cdot)$ *is continuous at* $\bar{p}$*;*

(c) $G(\bar{p})$ *is a singleton and* $\varphi(\cdot)$ *is locally Lipschitz at* $\bar{p}$*; and*

(d) $G(\bar{p})$ *is a singleton and* $\varphi(\cdot)$ *is continuous at* $\bar{p}$.

3.2. First-order directional differentiability

For $\bar{x} \in G(p)$, denote by $\Lambda(\bar{x}, p)$ the set of all Lagrange multipliers corresponding to $\bar{x}$.

We consider the following assumption
Assumption $(\mathbf{A}_4)$ *For every $t_k \downarrow 0$, for every $x^k \in G(p + t_k p^0)$ satisfying $x^k \to \bar{x} \in G(p)$, and for every $\lambda \in \Lambda(\bar{x}, p)$, the following inequality holds*

$$\liminf_{k \to +\infty} \frac{(x^k - \bar{x})^T \left(Q + \sum_{i \in I(\bar{x},p)} \lambda_i Q_i \right) (x^k - \bar{x})}{t_k} \geq 0.$$

Remark 3.1. *If $\nabla^2_{xx} L(\bar{x}, p, \lambda) = Q + \sum_{i \in I(\bar{x},p)} \lambda_i Q_i$ is positive semidefinite matrix, then (A_4) holds. In some cases, (A_4) is weaker than $(SOSC)_{p^0}$ in Auslender and Cominetti (1990) and the assumption of the calmness of global solution mapping in Minchenko and Tarakanov (2015) applied for $(QP(p))$; in some cases, (A_4) is also weaker than $(H3)$ in Minchenko and Sakochik (1996) applied for $(QP(p))$.*

Theorem 3.3. *If the problem $(QP(p))$ satisfies (SCQ), (A_3), and (A_4), then φ is first-order directional differentiable at p in every direction $p^0 \in \mathbb{P}$ and*

$$\begin{aligned} \varphi'(p, p^0) &= \min_{\bar{y} \in G(p)} \max_{\lambda \in \Lambda(\bar{y},p)} \left\{ f(\bar{y}, p^0) + \sum_{i=1}^{m} \lambda_i g_i(\bar{y}, p_i^0) \right\} \\ &= \min_{\bar{y} \in G(p)} \min_{h \in D(\bar{y},p,p^0)} \left\{ (Q\bar{y} + q)^T h + f(\bar{y}, p^0) \right\}. \end{aligned}$$

Theorem 3.4. *If the problem $(QP(p))$ satisfies (A_3), then φ is first-order directional differentiable at p in every direction $p^0 = (Q^0, q^0, 0) \in \mathbb{P}$ and*

$$\varphi'(p, p^0) = \min_{\bar{y} \in G(p)} \left\{ \frac{1}{2} \bar{y}^T Q^0 \bar{y} + (q^0)^T \bar{y} \right\}.$$

3.3. Second-order directional differentiability

Firstly, we consider the following assumption:

Assumption $(\mathbf{A}_5)$ *For every sequence $\{t_k\}, t_k \downarrow 0$, for every sequence $\{x^k\}$ satisfying $x^k \in G(p + t_k p^0)$, $x^k \to \bar{x} \in G(p)$, $h^k := (x^k - \bar{x})/t_k$, there exists $\bar{\lambda} \in \Lambda^*(\bar{x}, p)$*

such that

$$\liminf_{k\to\infty}(h^k, p^0)^T\nabla^2_{(x,p)}L(\bar{x}, p, \bar{\lambda})(h^k, p^0)$$
$$\geq \inf_{h\in D^*(\bar{x},p,p^0)} \max_{\lambda\in\Lambda^*(\bar{x},p)} (h, p^0)^T\nabla^2_{(x,p)}L(\bar{x}, p, \lambda)(h, p^0).$$

In some cases, (A_5) is also weaker than $(SOSC)_{p^0}$ and the assumption of the calmness of global solution mapping applied for $(QP(p))$.

For each $h \in D(\bar{x}, p, p^0)$, let

$$I(\bar{x}, h, p, p^0) := \{i \in I(\bar{x}, p) : (Q_i\bar{x} + q_i)^T h + g_i(\bar{x}, p^0) = 0\}.$$

We consider the following proposition.

Proposition 3.1. *Assume that, for every sequence $\{t_k\}, t_k \downarrow 0$, for every sequence $\{x^k\}, x^k \in G(p + t_kp^0)$, $x^k \to \bar{x} \in G(p)$, for every sequence $h^k := (x^k - \bar{x})/t_k$ satisfying $\|h^k\| \to \infty$, the following two conditions is satisfied:*

(b_1) $(Q_i^0\bar{x} + q_i^0)^T(x^k - \bar{x}) \geq 0$ *for every* $i \in I(\bar{x}, p)$*;*

(b_2) $(Q\bar{x}+q)^Tv \geq 0$ $\forall v \in \{u \in \mathbb{R}^n : (Q_i\bar{x}+q_i)^Tu \leq 0, i \in I(\bar{x}, h^k, p, p^0)\}$ *for every k large enough.*

Then, (A_5) holds.

The main result in thi section is presented as follows.

Theorem 3.5. *Consider the problem $(QP(p))$ and $p^0 \in \mathbb{P}$. If (SCQ) and (A_3)–(A_5) are satisfied, then φ is second-order directional differentiable at p in direction p^0 and*

$$\varphi''(p, p^0) = \min_{\bar{x}\in G(p,p^0)} \inf_{h\in D^*(\bar{x},p,p^0)} \max_{\lambda\in\Lambda^*(\bar{x},p)} \Bigg\{h^T\Bigg(Q + \sum_{i\in I(\bar{x},p)} \lambda_iQ_i\Bigg)h + +2\Bigg(Q^0\bar{x} + q^0 + \sum_{i\in I(\bar{x},p)} \lambda_i(Q_i^0\bar{x} + q_i^0)\Bigg)^T h\Bigg\}. \tag{3.31}$$

Corollary 3.1. *Consider the problem $(QP(p))$ and $p^0 \in \mathbb{P}$. Assume that (SCQ), (A_3), and at least one of the following conditions is satisfied:*
i) (A_4) and the assumptions of Proposition 3.1 hold;
ii) $(SOSC)_{p^0}$ holds at $\bar{x} \in G(p)$;
iii) $G(\cdot)$ is calm at $(p, \bar{x}) \in \mathbb{P} \times G(p)$.
Then, φ is second-order directional differentiable at p in direction p^0 and (3.31) holds.

The following result gives a sufficient condition for second-order directional differentiability of the optimal value function.

Theorem 3.6. *Assume that the problem $(QP(p))$ satisfies (SCQ) and (A_3), and φ is first-order directional differentiable at p in every direction $p^0 \in \mathbb{P}$. Assume that there exist $\bar{\lambda} \in \Lambda(\bar{x}, p)$ and, for every $t \downarrow 0$, $x_t \in G(p + tp^0), x_t \to \bar{x} \in G(p)$ such that*

$$\lim_{t\downarrow 0}(h_t, p^0)^T \nabla^2_{(x,p)} L(\bar{x}, p, \bar{\lambda})(h_t, p^0)$$

exists, where $h_t = (x_t - \bar{x})/t$, and

$$\lim_{t\downarrow 0} \frac{\bar{\lambda}_i g_i(x_t, p + tp^0)}{t^2} = 0.$$

Then φ is second-order directional differentiable at p in direction p^0 and

$$\varphi''(p, p^0) = \lim_{t\downarrow 0} \Bigg[h_t^T \Bigg(Q + \sum_{i\in I(\bar{x},p)} \bar{\lambda}_i Q_i \Bigg) h_t + \\ + 2\Bigg(Q^0\bar{x} + q^0 + \sum_{i\in I(\bar{x},p)} \bar{\lambda}_i (Q_i^0 \bar{x} + q_i^0) \Bigg)^T h_t \Bigg].$$

Chapter 4

Stability for extended trust region subproblems

This chapter devotes detailed discussion to a class of QCQP problems. Namely, we study stability and Lipschitzian stability for parametric extended trust region subproblems (ETRS). The material of this chapter is taken from [3,5,6].

4.1. Problem statement

In this section, we concern to parametric ETRS as follows

$$\begin{aligned} &\min\ f(x,Q,c) := \tfrac{1}{2}x^TQx + c^Tx \\ &\text{s.t. } x \in \mathbb{R}^n : x^TDx \le r,\ Ax + b \le 0, \end{aligned} \tag{$ET_m(w)$}$$

where $Q, D \in \mathbb{R}^{n\times n}$ are symmetric, D is positive definite, $c \in \mathbb{R}^n$, $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$, $r > 0$ and $w := (Q, c, D, r, A, b)$.

4.2. Some stability results for parametric ETRS

4.2.1. Continuity of the stationary solution map

The necessary condition for the lower semicontinuity of the multifunction $S(\bar{Q}, ., \bar{D}, \bar{r}, \bar{A}, .)$ is characterized in the following theorem.

Theorem 4.1. *Consider the problem $(ET_m(w))$ and $\bar{w} \in \mathbb{W}$. If $\bar{A}$ has full rank and $S(\bar{Q}, ., \bar{D}, \bar{r}, \bar{A}, .)$ is lower semicontinuous at $(\bar{q}, \bar{b})$, then $(ET_m(\bar{w}))$ satisfies (SCQ) and $S(\bar{w})$ is a nonempty set which contains at most 2^m points.*

Corollary 4.1. *Consider the problem $(ET_m(w))$ and $\bar{w} \in \mathbb{W}$. If $\bar{A}$ has full rank and $S(\cdot)$ is lower semicontinuous at $\bar{w}$ then $(ET_m(\bar{w}))$ satisfies (SCQ) and $S(\bar{w})$ is a nonempty set which contains at most 2^m points.*

The following result show some sufficient conditions for the semicontinuity of $S(\cdot)$.

Theorem 4.2. *Consider $(ET_m(w))$ and $\bar{w} \in \mathbb{W}$. If $S(\bar{w}) \neq \emptyset$ and at least one of the following conditions is satisfied:*

(i) $\bar{Q}+\lambda\bar{D}$ *is positive definite for every KKT pair (x, λ, μ) and $(ET_m(\bar{w}))$ satisfies (SCQ);*

(ii) $S(\bar{w})$ *is a singleton and $(ET_m(\bar{w}))$ satisfies (SCQ);*

(iii) $S(\bar{w})$ *is a singleton and φ is continuous at $\bar{w}$;*

(iv) $G(\cdot)$ *is lower semicontinuous at $\bar{w}$;*

(v) $S(\bar{w})$ *is finite and $S(\bar{w}) \cap \partial\mathcal{F}(\bar{w}) = \emptyset$;*

(vi) $\bar{Q}$ *is nonsingular and $S(\bar{w}) \cap \partial\mathcal{F}(\bar{w}) = \emptyset$,*

then $S(\cdot)$ is lower semicontinuous at $\bar{w}$.

Corollary 4.2. *Consider $(ET_m(w))$ and $\bar{w} \in \mathbb{W}$. If $(ET_m(\bar{w}))$ satisfies (SCQ) and $\bar{Q}$ is positive definite, then $S(\cdot)$ is lower semicontinuous at $\bar{p}$.*

4.2.2. Continuity of the optimal value function

The main result in this subsection is presented in the following theorem.

Theorem 4.3. *Consider the problem $(ET_m(w))$ and $\bar{w} \in \mathbb{W}$. The following assertions hold:*

(i) *φ is lower semicontinuous at $\bar{w}$;*

(ii) *φ is upper semicontinuous at $\bar{w}$ if $(ET_m(\bar{w}))$ satisfies (SCQ);*

(iii) *φ is continuous at $\bar{w}$ if $(ET_m(\bar{w}))$ satisfies (SCQ);*

(iv) *If $\mathcal{F}(\bar{w})$ is nonempty and if φ is continuous at $\bar{w}$, then $(ET_m(\bar{w}))$ satisfies (SCQ);*

(v) *If $\mathcal{F}(\bar{w})$ is empty, then φ is continuous at $\bar{w}$.*

4.3. ETRS with a linear inequality constraint

4.3.1. Lower semicontinuity of the stationary solution map

A necessary and sufficient conditions for the lower semicontinuity of the stationary solution map are proposed below.

Theorem 4.4. *Consider the problem* $(ET_1(w))$ *and* $\bar{w} \in \mathbb{W}$. *The multifunction* $S(\bar{Q}, ., \bar{r}, \bar{a}, .)$ *is lower semicontinuous at* $(\bar{q}, \bar{b})$ *if and only if* $(ET_1(\bar{w}))$ *satisfies* (SCQ) *and* $S(\bar{w})$ *is a singleton.*

Theorem 4.5. *Consider the problem* $(ET_1(w))$ *and* $\bar{w} \in \mathbb{W}$. *The multifunction* $\tilde{w} \mapsto S(\tilde{w})$ *is lower semicontinuous at* $\bar{p}$ *if and only if* $(ET_1(\bar{w}))$ *satisfies (SCQ) and* $S(\bar{w})$ *is a singleton.*

Corollary 4.3. *Consider the problem* $(ET_1(w))$ *and* $\bar{w} \in \mathbb{W}$. *Assume that the problem* $(ET_1(\bar{w}))$ *satisfies (SCQ). If* $S(\cdot)$ *is continuous at* $\bar{w}$ *then* $G(\cdot)$ *is continuous at* $\bar{w}$.

4.3.2. Coderivatives of the normal cone mapping

The feasible region of the problem $(ET_1(\bar{w}))$ is rewritten as follows

$$\mathcal{F}(r, b) := \{x \in \mathbb{R}^n : \|x\| \le r, a^T x + b \le 0\},$$

which depends on the parameter (r, b).

Denote

$$N(x; \mathcal{F}(r, b)) := \{v \in \mathbb{R}^n : \langle v, y - x\rangle \le 0 \quad \forall y \in \mathcal{F}(r, b)\}$$

be *the normal cone* to the convex set $\mathcal{F}(r, b)$ at x.

It is easy to see that

$$N(x; \mathcal{F}(r, b)) = \begin{cases} \{0\} & \text{if } \|x\| < r, a^T x + b < 0, \\ \{\theta x : \theta \ge 0\} & \text{if } \|x\| = r, a^T x + b < 0, \\ \{\gamma a : \gamma \ge 0\} & \text{if } \|x\| < r, a^T x + b = 0, \\ \{\theta x + \gamma a : \theta \ge 0, \gamma \ge 0\} & \text{if } \|x\| = r, a^T x + b = 0, \\ \emptyset & \text{if } \|x\| > r \text{ or } a^T x + b > 0. \end{cases}$$

For every $(x, r, b) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}$, we put

$$\mathcal{N}(x, r, b) = N(x; \mathcal{F}(r, b)).$$

If $r \leq 0$ then it is convenient to set $\mathcal{N}(x, r, b) = \emptyset$ for all $x \in \mathbb{R}^n$. Hence $\mathcal{N} : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \rightrightarrows \mathbb{R}^n$ is a multifunction with closed convex values and called be *the normal cone mapping related to parametric* $(ET_1(\bar{w}))$.

In this section, we calculate and estimate the Fréchet and Mordukhovich coderivatives of the normal cone mapping related to the parametric $(ET_1(\bar{w}))$.

Fix $\bar{w} := (\bar{x}, \bar{r}, \bar{b}, \bar{v}) \in gph\mathcal{N}$, we compute and estimate the Fréchet coderivative of the normal cone mapping.

Theorem 4.6. *For every $\bar{w} = (\bar{x}, \bar{r}, \bar{b}, \bar{v}) \in gph\mathcal{N}$, the assertions are valid:*

(a) *If $\|\bar{x}\| < \bar{r}$ and $a^T\bar{x} + \bar{b} < 0$, then $\bar{v} = 0$ and*

$$\widehat{D}^*\mathcal{N}(\bar{w})(v^*) = \{(0_{\mathbb{R}^n}, 0_{\mathbb{R}}, 0_{\mathbb{R}})\};$$

(b) *If $\|\bar{x}\| = \bar{r}, a^T\bar{x} + \bar{b} < 0,$ and $\bar{v} = \theta\bar{x}$ with $\theta > 0$ then*

$$\widehat{D}^*\mathcal{N}(\bar{w})(v^*) = \begin{cases} \Omega_1(\bar{w})(v^*) & \text{if } \langle v^*, \bar{x}\rangle = 0, \\ \emptyset & \text{if } \langle v^*, \bar{x}\rangle \neq 0; \end{cases}$$

(c) *If $\|\bar{x}\| = \bar{r}, a^T\bar{x} + \bar{b} < 0,$ and $\bar{v} = 0$ then*

$$\widehat{D}^*\mathcal{N}(\bar{w})(v^*) = \begin{cases} \Omega_2(\bar{w})(v^*) & \text{if } \langle v^*, \bar{x}\rangle \geq 0, \\ \emptyset & \text{if } \langle v^*, \bar{x}\rangle < 0; \end{cases}$$

(d) *If $\|\bar{x}\| < \bar{r}, a^T\bar{x} + \bar{b} = 0,$ and $\bar{v} = \gamma a$ with $\gamma > 0$ then*

$$\widehat{D}^*\mathcal{N}(\bar{w})(v^*) = \begin{cases} \Omega_3(\bar{w})(v^*) & \text{if } \langle v^*, a\rangle = 0, \\ \emptyset & \text{if } \langle v^*, a\rangle \neq 0; \end{cases}$$

(e) *If $\|\bar{x}\| < \bar{r}, a^T\bar{x} + \bar{b} = 0,$ and $\bar{v} = 0$ then*

$$\widehat{D}^*\mathcal{N}(\bar{w})(v^*) = \begin{cases} \Omega_4(\bar{w})(v^*) & \text{if } \langle v^*, a\rangle \geq 0, \\ \emptyset & \text{if } \langle v^*, a\rangle < 0; \end{cases}$$

(f) *If $\|\bar{x}\| = \bar{r}, a^T\bar{x} + \bar{b} = 0,$ and $\bar{v} = \theta\bar{x} + \gamma a$ with $\theta > 0, \gamma > 0$ then*

$$\widehat{D}^*\mathcal{N}(\bar{w})(v^*) \subset \begin{cases} \Omega_5(\bar{w})(v^*) & \text{if } \langle v^*, \bar{x}\rangle = 0 \text{ and } \langle v^*, a\rangle = 0, \\ \emptyset & \text{otherwise} \end{cases}$$

(g) *If* $\|\bar{x}\| = \bar{r}, a^T\bar{x} + \bar{b} = 0,$ *and* $\bar{v} = \theta\bar{x}$ *with* $\theta > 0,$ *then*

$$\widehat{D}^*\mathcal{N}(\bar{w})(v^*) \subset \begin{cases} \Omega_5^1(\bar{w})(v^*) & \textit{if } \langle v^*, \bar{x}\rangle = 0 \textit{ and } \langle v^*, a\rangle \geq 0, \\ \emptyset & \textit{otherwise} \end{cases}$$

(h) *If* $\|\bar{x}\| = \bar{r}, a^T\bar{x} + \bar{b} = 0,$ *and* $\bar{v} = \gamma a$ *with* $\gamma > 0,$ *then*

$$\widehat{D}^*\mathcal{N}(\bar{w})(v^*) \subset \begin{cases} \Omega_5^2(\bar{w})(v^*) & \textit{if } \langle v^*, \bar{x}\rangle = 0 \textit{ and } \langle v^*, \bar{x}\rangle \geq 0, \\ \emptyset & \textit{otherwise} \end{cases}$$

(i) *If* $\|\bar{x}\| = \bar{r}, a^T\bar{x} + \bar{b} = 0,$ *and* $\bar{v} = 0$ *then*

$$\widehat{D}^*\mathcal{N}(\bar{w})(v^*) \subset \begin{cases} \Omega_6(\bar{w})(v^*) & \textit{if } \langle v^*, \bar{x}\rangle \geq 0, \quad \textit{and } \langle v^*, a\rangle \geq 0, \\ \emptyset & \textit{otherwise.} \end{cases}$$

Theorem 4.7. *For every* $\bar{w} = (\bar{x}, \bar{r}, \bar{b}, \bar{v}) \in gph\mathcal{N}$, *the assertions are valid:*

(a) *If* $\|\bar{x}\| < \bar{r}$ *and* $a^T\bar{x} + \bar{b} < 0,$ *then* $\bar{v} = 0$ *and*

$$D^*\mathcal{N}(\bar{w})(v^*) = \{(0_{\mathbb{R}^n}, 0_{\mathbb{R}}, 0_{\mathbb{R}})\};$$

(b) *If* $\|\bar{x}\| = \bar{r}, a^T\bar{x} + \bar{b} < 0,$ *and* $\bar{v} = \theta\bar{x}$ *with* $\theta > 0$ *then*

$$D^*\mathcal{N}(\bar{w})(v^*) = \begin{cases} \Omega_1(\bar{w})(v^*) & \textit{if } \langle v^*, \bar{x}\rangle = 0, \\ \emptyset & \textit{if } \langle v^*, \bar{x}\rangle \neq 0; \end{cases}$$

(c) *If* $\|\bar{x}\| = \bar{r}, a^T\bar{x} + \bar{b} < 0,$ *and* $\bar{v} = 0$ *then*

$$D^*\mathcal{N}(\bar{w})(v^*) = \begin{cases} \{0_{\mathbb{R}^{n+2}}\} & \textit{if } \langle v^*, \bar{x}\rangle < 0, \\ \Omega_2(\bar{w})(v^*) & \textit{if } \langle v^*, \bar{x}\rangle > 0, \\ \Omega_2'(\bar{w})(v^*) & \textit{if } \langle v^*, \bar{x}\rangle = 0; \end{cases}$$

(d) *If* $\|\bar{x}\| < \bar{r}, a^T\bar{x} + \bar{b} = 0,$ *and* $\bar{v} = \gamma a$ *with* $\gamma > 0$ *then*

$$D^*\mathcal{N}(\bar{w})(v^*) = \begin{cases} \Omega_3(\bar{w})(v^*) & \textit{if } \langle v^*, a\rangle = 0, \\ \emptyset & \textit{if } \langle v^*, a\rangle \neq 0; \end{cases}$$

(e) *If* $\|\bar{x}\| < \bar{r}, a^T\bar{x} + \bar{b} = 0,$ *and* $\bar{v} = 0$ *then*

$$D^*\mathcal{N}(\bar{w})(v^*) = \begin{cases} \{0_{\mathbb{R}^{n+2}}\} & \textit{if } \langle v^*, a\rangle < 0, \\ \Omega_4(\bar{w})(v^*) & \textit{if } \langle v^*, a\rangle > 0, \\ \Omega_3(\bar{w})(v^*) & \textit{if } \langle v^*, a\rangle = 0; \end{cases}$$

(f) *If* $\|\bar{x}\| = \bar{r}, a^T\bar{x} + \bar{b} = 0,$ *and* $\bar{v} = \theta\bar{x} + \gamma a$ *with* $\theta > 0, \gamma > 0$ *then*

$$D^*\mathcal{N}(\bar{w})(v^*) \subset \begin{cases} \Omega_5(\bar{w})(v^*) & \text{if } \langle v^*, \bar{v}\rangle = 0, \\ \emptyset & \text{if } \langle v^*, \bar{v}\rangle \neq 0; \end{cases}$$

(g) *If* $\|\bar{x}\| = \bar{r}, a^T\bar{x} + \bar{b} = 0,$ *and* $\bar{v} = \theta\bar{x}$ *with* $\theta > 0$ *then*

$$D^*\mathcal{N}(\bar{w})(v^*) \subset \begin{cases} \Omega_5(\bar{w})(v^*) \cup \Omega_1(\bar{w})(v^*) & \text{if } \langle v^*, \bar{x}\rangle = 0, \\ \emptyset & \text{if } \langle v^*, \bar{x}\rangle \neq 0; \end{cases}$$

(h) *If* $\|\bar{x}\| = \bar{r}, a^T\bar{x} + \bar{b} = 0,$ *and* $\bar{v} = \gamma a$ *with* $\gamma > 0$ *then*

$$D^*\mathcal{N}(\bar{w})(v^*) \subset \begin{cases} \Omega_5(\bar{w})(v^*) \cup \Omega_3(\bar{w})(v^*) & \text{if } \langle v^*, a\rangle = 0, \\ \emptyset & \text{if } \langle v^*, a\rangle \neq 0; \end{cases}$$

(i) *If* $\|\bar{x}\| = \bar{r},$ $a^T\bar{x} + \bar{b} = 0$ *and* $\bar{v} = 0$ *then*

$$D^*\mathcal{N}(\bar{w})(v^*) = \begin{cases} \{(0_{\mathbb{R}^n}, 0_{\mathbb{R}}, 0_{\mathbb{R}})\} & \text{if } \langle v^*, \bar{x}\rangle < 0 \ \text{ and } \langle v^*, a\rangle < 0, \\ \Omega_4(\bar{w})(v^*) & \text{if } \langle v^*, \bar{x}\rangle < 0 \ \text{ and } \langle v^*, a\rangle > 0, \\ \Omega_2(\bar{w})(v^*) & \text{if } \langle v^*, \bar{x}\rangle > 0 \ \text{ and } \langle v^*, a\rangle < 0, \\ \Omega_2'(\bar{w})(v^*) & \text{if } \langle v^*, \bar{x}\rangle = 0 \ \text{ and } \langle v^*, a\rangle < 0, \\ \Omega_3(\bar{w})(v^*) & \text{if } \langle v^*, \bar{x}\rangle < 0 \ \text{ and } \langle v^*, a\rangle = 0; \end{cases}$$

and

$$D^*\mathcal{N}(\bar{w})(v^*) \subset \begin{cases} \Omega_7(\bar{w})(v^*) & \text{if } \langle v^*, \bar{x}\rangle > 0 \ \text{ and } \langle v^*, a\rangle > 0, \\ \Omega_8(\bar{w})(v^*) & \text{if } \langle v^*, \bar{x}\rangle = 0 \ \text{ and } \langle v^*, a\rangle > 0, \\ \Omega_9(\bar{w})(v^*) & \text{if } \langle v^*, \bar{x}\rangle > 0 \ \text{ and } \langle v^*, a\rangle = 0, \\ \Omega_{10}(\bar{w})(v^*) & \text{if } \langle v^*, \bar{x}\rangle = 0 \ \text{ and } \langle v^*, a\rangle = 0, \end{cases}$$

where

$$\Omega_1(\bar{\omega})(v^*) := \{(x^*, r^*, b^*) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} : b^* = 0,\ x^* = -\frac{r^*}{\bar{r}}\bar{x} + \theta v^*\},$$

$$\Omega_2(\bar{\omega})(v^*) := \{(x^*, r^*, b^*) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} : b^* = 0,\ r^* \le 0,\ x^* = -\frac{r^*}{\bar{r}}\bar{x}\},$$

$$\Omega_2'(\bar{\omega})(v^*) := \{(x^*, r^*, b^*) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} : b^* = 0,\ x^* = -\frac{r^*}{\bar{r}}\bar{x}\},$$

$$\Omega_3(\bar{\omega})(v^*) := \{(x^*, r^*, b^*) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} : r^* = 0,\ x^* = b^* a\},$$

$$\Omega_4(\bar{\omega})(v^*) := \{(x^*, r^*, b^*) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} : r^* = 0,\ x^* = b^* a,\ b^* \ge 0\},$$

$$\Omega_5(\bar{\omega})(v^*) := \{(x^*, r^*, b^*) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} : \langle x^*, \bar{x}\rangle + r^*\bar{r} + b^*\bar{b} = 0\},$$

$$\Omega_5^1(\bar{\omega})(v^*) := \{(x^*, r^*, b^*) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} : \langle x^*, \bar{x}\rangle + r^*\bar{r} + b^*\bar{b} = 0,$$
$$\langle v^*, a\rangle \ge 0,\ b^* \ge 0\},$$

$$\Omega_5^2(\bar{\omega})(v^*) := \{(x^*, r^*, b^*) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} : \langle x^*, \bar{x}\rangle + r^*\bar{r} + b^*\bar{b} = 0,$$

$$\Omega_6(\bar{\omega})(v^*) := \{(x^*, r^*, b^*) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} : \langle x^*, \bar{x}\rangle + r^*\bar{r} + b^*\bar{b} = 0,$$
$$x^* \in \mathrm{pos}\{\bar{x}, a\},\ b^* \ge 0,\ r^* \le 0\},$$

$$\Omega_7(\bar{\omega}) = \Omega_2(\bar{\omega})(v^*) \cup \Omega_4(\bar{\omega})(v^*) \cup \Omega_6(\bar{\omega})(v^*),$$

$$\Omega_8(\bar{\omega}) = \Omega_2'(\bar{\omega})(v^*) \cup \Omega_4(\bar{\omega})(v^*) \cup \Omega_5^1(\bar{\omega})(v^*) \cup \Omega_6(\bar{\omega})(v^*),$$

$$\Omega_9(\bar{\omega}) = \Omega_2(\bar{\omega})(v^*) \cup \Omega_3(\bar{\omega})(v^*) \cup \Omega_5^2(\bar{\omega})(v^*) \cup \Omega_6(\bar{\omega})(v^*),$$

$$\Omega_{10}(\bar{\omega}) = \Omega_2'(\bar{\omega})(v^*) \cup \Omega_3(\bar{\omega})(v^*) \cup \Omega_5(\bar{\omega})(v^*) \cup \Omega_6(\bar{\omega})(v^*).$$

4.3.3. Lipschitzian stability

In this subsection, we use obtained results and the Mordukhovich criterion (see Mordukhovich (2006)) for the locally Lipschitz-like property of multifunctions to investigate Lipschitzian stability of $(ET_1(\bar{w}))$ with respect to the linear perturbations. We always assume that $(ET_1(\bar{w}))$ satisfies LICQ.

The stationary solution set of $(ET_1(w))$ is rewritten by $S(Q, q, r, b)$. Recall that (see, for instance, Facchinei and Pang (2003)), under LICQ, x is a stationary solution of $(ET_1(\bar{w}))$ if and only if

$$\langle Qx + q, y - x\rangle \ge 0 \quad \forall y \in \mathcal{F}(r, b),$$

i.e., x is a global solution of the generalized equation

$$0 \in Qx + q + N(x; \mathcal{F}(r, b)).$$

We have

$$y \in H(x, z) + M(x, z),$$

where $y := -q, z := (Q, r, b), H(x, z) := Qx$ and $M(x, z) := \mathcal{N}(x, r, b)$.

Denote by $\mathbb{R}_s^{n\times n}$ the linear subspace of symmetric $n \times n$ matrices in $\mathbb{R}^{n\times n}$ and put $Z := \mathbb{R}_s^{n\times n} \times \mathbb{R} \times \mathbb{R}$. Then, $S(\cdot)$ can be interpreted as the multifunction $\widetilde{S} : Z \times \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ defined by

$$\widetilde{S}(z, y) = \{x \in \mathbb{R}^n : y \in H(x, z) + M(x, z)\}.$$

Then, we have

$$\widetilde{S}(z, y) = S(Q, q, r, b).$$

The following theorem estimates the Mordukhovich coderivative of $\widetilde{S}(\cdot)$.

Theorem 4.8. *Consider the problem $(ET_1(\bar{w}))$ and $(\bar{z}, \bar{y}, \bar{x}) \in gph\widetilde{S}$. For each $x^* \in \mathbb{R}^n$, if $(y^*, z^*) \in D^*\widetilde{S}(\bar{z}, \bar{y}, \bar{x})(x^*)$ then:*

$$\begin{aligned}
&\bar{Q}y^* = 2x^*,\\
&Q^*_{ij} = -y^*_i \bar{x}_j,\\
&(x^*, r^*, b^*) \in D^*\mathcal{N}(\bar{x}, \bar{r}, \bar{b}, \bar{v})(-y^*);
\end{aligned}$$

where $\bar{z} = (\bar{Q}, \bar{r}, \bar{b})$, $\bar{v} = \bar{y} - H(\bar{x}, \bar{z}) = -\bar{q} - \bar{Q}\bar{x}$, $z^* = (Q^*, r^*, b^*)$ and Q^*_{ij} is the (i, j)th element of Q^*.

Some sufficient conditions for the local Lipschitz-like property of $S(\cdot)$ is estimated as follows.

Theorem 4.9. *The multifunction $(\widetilde{Q}, \widetilde{q}, \widetilde{r}, \widetilde{b}) \mapsto S(\widetilde{Q}, \widetilde{q}, \widetilde{r}, \widetilde{b})$ is locally Lipschitz-like around $(\bar{Q}, \bar{q}, \bar{r}, \bar{b}, \bar{x}) \in gphS$ if at least one of the following conditions is satisfied:*

(i) $\|\bar{x}\| < \bar{r}, a^T\bar{x} + \bar{b} < 0$ *and* $det\bar{Q} \neq 0$*;*

(ii) $\|\bar{x}\| = \bar{r}, a^T\bar{x} + \bar{b} < 0$ *and* $\bar{Q}\bar{x} + \bar{q} = \theta\bar{x}, \theta > 0$;

(iii) $\|\bar{x}\| = \bar{r}, a^T\bar{x} + \bar{b} < 0$, $\bar{Q}\bar{x} + \bar{q} = 0$, $rank(\bar{Q}; \bar{x}) = n$ *and* $\langle \bar{x}, u\rangle = 0$ *for every* $u \in Null(\bar{Q})$*;*

(iv) $\|\bar{x}\| < \bar{r}, a^T\bar{x} + \bar{b} = 0$, $\bar{Q}\bar{x} + \bar{q} = \gamma a, \gamma > 0$, *and* $rank(\bar{Q}; a) = n$*;*

(v) $\|\bar{x}\| < \bar{r}, a^T\bar{x} + \bar{b} = 0$, $\bar{Q}\bar{x} + \bar{q} = 0$, $rank(\bar{Q}; a) = n$ *and* $\langle a, u\rangle = 0$ *for every* $u \in Null(\bar{Q})$*;*

(vi) $\|\bar{x}\| = \bar{r}, a^T\bar{x} + b = 0$, *$b$ is unperturbed and* $det\bar{Q} \neq 0$.

General Conclusions

Our main results for the parametric quadratic programming problems with non-convex objective function include:

1. A Frank-Wolfe type theorem and an Eaves type theorem for solution existence;
2. Conditions for upper and lower semicontinuities of the global and local optimal solution map;
3. Some stability results for the stationary solution set;
4. Conditions for the continuity, Lipschitz property and directional differentiability of the optimal value function;
5. Upper estimations for the Mordukhovich coderivative and conditions for the local Lipschitz-like property of the stationary solution map in parametric extended trust region subproblems.

List of Author's Papers

1. Nghi, T.V., Tam, N.N.: Continuity and directional differentiability of the optimal value function in parametric quadratically constrained nonconvex quadratic programs, *Acta Math. Vietnam.*, 2017, 42(2), 311–336 (**SCOPUS**)

2. Tam, N.N., Nghi, T.V.: On the solution existence and stability of quadratically constrained nonconvex quadratic programs, *Optim. Lett.*, 2017, 1–19, DOI: 10.1007/s11590-017-1163-4 (**SCIE**)

3. Nghi, T.V.: Coderivatives related to parametric extended trust region subproblem and their applications, *Taiwanese J. Math.*, 2017, 1–27, DOI:10.11650/tjm/170907 (**SCI**)

4. Nghi, T.V.: On stability of solutions to parametric generalized affine variational inequalities, *Optimization*, 2017, 1–17, DOI:10.1080/02331934.2017.1394297 (**SCIE**)

5. Nghi, T.V., Tam, N.N.: Stability of the Karush-Kuhn-Tucker point set in parametric extended trust region subproblems (submitted to Acta Math. Vietnam.)

6. Nghi, T.V., Tam, N.N.: Stability for parametric extended trust region subproblems (submitted to Pac. J. Optim.)